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The Mosses of Bengal.

First Contribution

By

Paul Brühl and Nagendranath Sarkar

Introductory Remarks.

By P. Brühl.

Our Knowledge of True Mosses (*Musci veri*) indigenous in large parts of India is yet very incomplete. This statement applies also to Bengal. Prof. Brotherus, in the second edition of Engler's "Pflanzenfamilien" mentions sixteen species as occurring in this province. A few of them were probably gathered in the Sikkim Terai or even adjacent parts of the lower Sikkim Himalaya. One of the species was probably accidentally introduced into the Royal Botanic Gardens, Sibpur. Additional species, widely distributed in tropical and subtropical regions, but not yet recorded from Bengal, will probably be found to be inhabitants of Bengal. An interesting case is that of *Hyophila involuta* (Hooker) Jaeger. This species is not mentioned by Prof. Brotherus as occurring in Bengal, but it grows plentifully on the garden wall of the Botanical Laboratory of the University Science College. The following species have been reported from Bengal according to the second edition of Engler's "Pflanzenfamilien:"

Fissidens titalayanus C. Müller,
Fissidens Kurzii C. Müller,
Fissidens terraicola C. Müller,
Fissidens auriculatus C. Müller,
Fissidens subpalmatus C. Müller,
Calymperes tenerum C. Müller,
Barbula gangetica C. Müller,
Barbula comosa Dozy et Molkenboer,
Barbula orientalis (Willdenow) Brotherus,

Splachnobryum indicum C. Müller,
Tortula Kurzii C. Müller,
Tortula muralis (Lin) Hedwig,
Bryum coronatum Schwaegrichen,
Pinatella calcuttensis (C. Müller) Fleischer,
Taxithelium nepalense (Schwaegrichen) Brotherus.
Callicostella papillata (Montagne) Jaeger.

To these must be added

Fissidens bengalensis Hampe, mentioned in Sir David Prain's "The Vegetation of the Districts of Hughli-Howrah and the 24-Pergannahs" (*Records of the Botanical Survey of India*, Vol. III, No. 2, 1905).

Unfortunately many of the moss specimens in collections sent to herbaria are without capsules and their identification is rendered a matter of considerable difficulty. I have therefore worked out an analytical key to all those genera which have been reported from the whole of the area extending from the Caucasus to Tonkin and Formosa in one direction and from the Himalayas to the Eastern Islands of the Indian Archipelago in the other, based as far as possible on vegetative characters. The key will be accompanied by an enumeration of all the species reported from that area up to the present and will shortly be published in the Records of the Botanical Survey of India.

It is very unfortunate that Levier's exsiccata issued under the titles "Musci Indiae Orientalis, curante W. Gollan lecti," "Bryotheca exotica, cent. I," and the corresponding collections from Sikkim, Bhotan and Tenasserim as well Max Fleischer's Javanese exsiccata are not available in India. The descriptions of the species indigenous in the Indian Archipelago contained in Max Fleischer's "Die Moose dar Flora von Buitenzorg" are so excellent and detailed that the non-possession of the exsiccata are not so severely felt by Indian bryologists. It is very different as regards Levier's exsiccata. A large number of the species have been given names on the sheets, mostly by C. Müller and Prof. Brotherus, but have never been described, they are therefore *species ineditae*, and as far as India is concerned, the names are, strictly speaking, *nomina nuda* and may have to be treated as such, the more so, as notwithstanding my endeavours to secure a set in the European book-market, I have not been successful.

It is intended to work out and publish detailed descriptions of all the species found in Bengal together with illustrations of them. The present contribution is the first instalment. I have to thank my co-worker, Mr. Nagendranath Sarkar for the careful determination of the genera by the aid of my analytical key and the illustrations and measurements made with the help of a Leitz echelon-ocular-micrometer.

Description of Species

By P. B. and N. N. S.

Family : **Calymperaceae.**

Genus : **Calymperes.**

Species : **Calymperes noakhalensis.** Brühl et Sarkar, *sp. nova.*

Caule 10-15 mm. alto, dense folioso ; foliis normalibus erecto-patentibus, siccis recurvis vel flexuosis vel varie tortis, inferioribus oblanceolato-spatulatis, apice rotundatis, basin versus attenuatis, mediis atque superioribus anguste spatulatis lingulatisve, apice rotundatis, basin versus attenuatis, margine undulato-incurvis, minutissime serratis vel subintegerrimis, costa basi subcomplanata, superne semitereti, dorso convexa, aut paullo projecta aut apici laminae approximata, eurycystidibus sectione subquadratibus seriem unam completam, at hujus latere ventrali duobus tribusve seriem incompletam formantibus ; foliis abnormibus angustissimis, elongatis, costa apice gemmas parvas numerosas, multicellulares, ellipticis vel lineares gerente ; cellulis basalibus subrectangularibus, superioribus subquadratis atque irregulariter polygoniis, subisodiametricis, teniolis series tres ad duo formantibus ; sporogoniis ignotis.

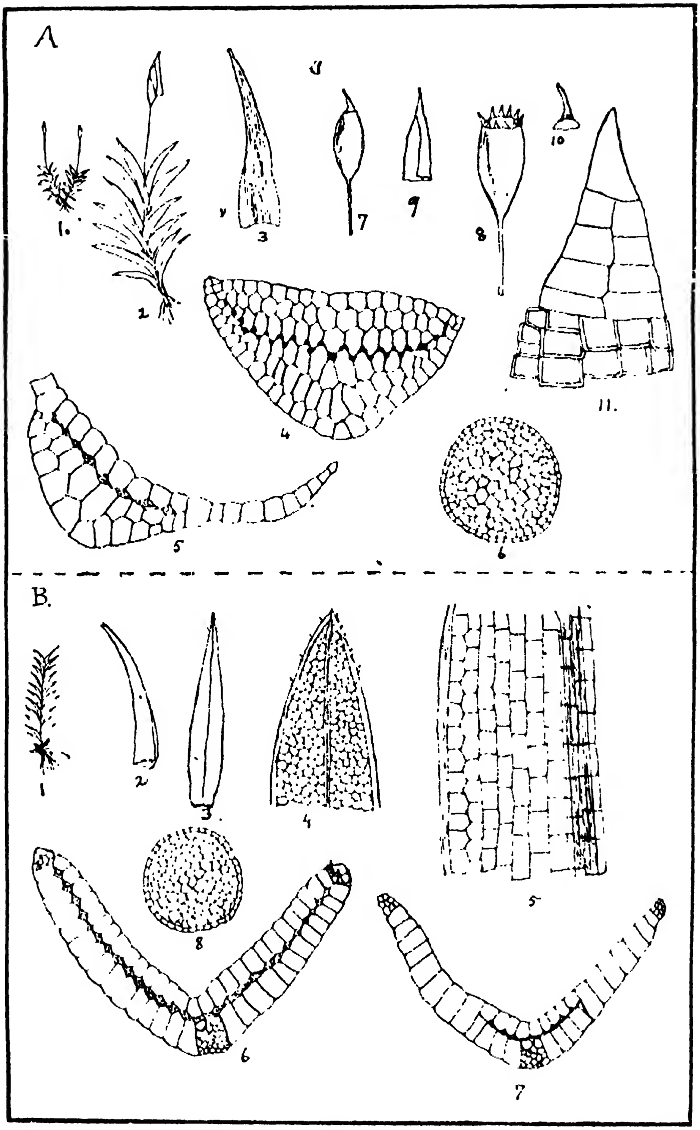
Found in cushions of *Octoblepharum albidum* Hedwig on the stems of Palm trees associated with *Leucophanes octoblepharoides* Bridel ; when dry reddish brown, 10-15 mm. high, densely foliose ; rhizoids arising from the base of the stem and the lower leaves, thin, smooth : central column absent, fundamental tissue in cross-section consisting of polygonal thin-walled cells, peripheral cells narrower and thin-walled ; leaves when moist erecto-patent, when dry recurved or flexuose or variously twisted, lower leaves oblanceolate-spatulate, at the apex submucronate due to the midrib slightly excurrent, sometimes

with a few minute obovoid brood-bodies, 2 mm. long, 0·5 mm. broad above the middle and 0·2 mm. near the base, middle and upper leaves narrow-spatulate or-lingulate, rounded at the apex, gradually narrowed down towards the base, 3·3-4 mm. long and 0·6-0·9 mm. broad at the apex and 0·3-0·4 mm. broad at the base; margin of the normal leaves minutely serrulate or nearly entire, undulately incurved, the sheathing part $\frac{1}{5}$ of the length of the whole leaf; abnormal leaves 5-6 mm. long, very narrow and 0·1 mm. broad at the apex, the laminar parts near their base of about the same breadth as the midrib, like the midrib narrowing down towards the apex, the midrib bearing a clvster of minute brood-bodies at its tip; midrib of the normal leaves rather flat at the base, semiterete upwards and dorsally convex; midrib of the normal leaves slightly projecting, approaching but not reaching quite up to the tips, deuter-cells nearly square in section, in a median row, usually accompanied on their ventral side by two, sometimes three similar deuter cells; cells on both the ventral and dorsal sides of the deuter-cells numerous, much narrow, the dorsal and ventral peripheral cells papillose; boundary between the cancelline and laminar area not step-like, cancellines in face view rectangular with the upper cross-walls more or less oblique, mostly 38-44 μ long, the uppermost cells distinctly shorter, their width gradually diminishing from the midrib outwards from 20 to 8 μ, laminar cells square or irregularly polygonal, subisodiametric, 8-12 μ in diameter, tenioles 30-40 μ long and 6-10 μ broad extending from above the base of the leaf to more than $\frac{3}{4}$ of its length in three, at the upper end in two rows; marginal cells in two rows somewhat longer than broad, those of the outer row projecting at their upper corner as minute teeth except those below the level of the origin of the teniole, outer wall concave; brood bodies of the abnormal leaves numerous, elliptic and sublinear, pointed, multicellular; sporogones unknown.

Our species differs from *C. molluccense* Schwaegrichen by the sheathing part passing gradually, not more or less abruptly, into the laminar parts of the leaves and the cancelline cells diminishing quite gradually in width from the midrib towards the teniole.

It differs from *C. Hampei* Dozy et Molkenboer by the peripheral cells of the stem forming a single layer and their cell-walls not being thickened and by the sheath gradually passing into the laminar part, not obovate and somewhat abruptly passing into the lamina; further by the normal leaves gradually attenuated from base to apex and not

PLATE I.



sublingulate, their midrib scarcely projecting beyond the lamina and not distinctly protruding and the tenioles never more than in 3 rows. *C. Hampei* is also distinguished by its less rigid habit and its white and shining leaf-basis.

C. gemmiphyllum Fleischer differs from our species in various characters ; it may however be noticed that it has a double row of deuter-cells, but as in our species the lower row is complete and the upper row very partial ; in *C. gemmiphyllum* it is the upper row which is complete, whilst the lower row is only partially developed.

Family : **Leucobryaceae.**

Sub-family : **Leucophanoideae.**

Genus : **Leucophanes.**

Species : **Leucophanes octoblepharoides** Bridel.

Forming soft whitish-green, rather dense cushions on the stems of Palm-trees ; stem erect, simple or sparsely branched, 4-20 mm. high, cross-section sub-circular, central column wanting; leaves subconduplicate, subcarinate, sheathing near their base, margins not touching, 0·3-0·6 mm. long, narrow-lanceolate, narrower at the base, acute or mucronate or sub-obtuse and somewhat remotely serrulate at the apex, consisting mainly of a pseudolamina, a dorsal multicellular narrow dorsally smooth roundish or slightly concave middle stereome forming a keel, and of a marginal seam consisting of several layers of hyaline exohyalocysts and extending right up to the apex ; the pseudolamina consisting for the most part of a dorsal and ventral layer of leucocysts and a single median layer of chlorocysts ; leucocysts rectangular in cross-section, the outer cell-walls somewhat convex or nearly plane, chlorocysts rhombic, in cross-section considerably smaller than the leucocysts, the layer of chlorocysts on either side of the keel, as seen in cross-section, arranged nearly in a straight line (not zigzag); the marginal cells and the stereids narrower than even the chlorocysts ; on the dorsal side of the chlorocysts next the median stereome a narrow layer of leucocysts ; the basal part of the pseudolamina consisting, besides the middle stereome and the marginal group of leucocysts, of two strata of leucocysts and a supra-median layer of chlorocysts, the remaining part of the pseudolamina consisting of a single layer of leucocysts rectangular in cross-section.

The specimens available were without sporogones. The following description of the sexual part is taken from Max Fleischer's "Laubmoosflora von Java." Perichaetial leaves slightly smaller than the stem-leaves; vaginula cylindric; seta straight, 6-8 mm. in height, reddish yellow, neck of the capsule distinct, with a row of stomata; capsule elongate-ovoid, orifice somewhat contracted; epidermal cells irregularly rectangular to elongate-pentagonal or -hexagonal; lid short-conical, nearly as long as the urn; calyptra hood-shaped, enveloping the capsule and fugacious; peristome inserted slightly below the rim; peristome-teeth 16, lanceolate, cross-bars very thin and not protruding; spores greenish, 12-16 μ in diameter, finely punctate.

The specimen described was found in October, growing on the stems of Palm-trees at Noakhali, Eastern Bengal. The species is widely distributed and extends from Nepal to the Pacific Islands.

See—Engler's "Die natürlichen Pflanzenfamilien," second edition, Vol. X, pp. 224-225, and Fleischer's "Die Musci der Flora von Buitenzorg," Vol. I, pp. 174-176.

Subfamily: **Octoblepharoideae.**

Genus: **Octoblepharum.**

Species: **Octoblepharum albidum** Hedwig.

Plant-mass forming dense, irregular, greenish-white mats on the bark of trees, chiefly on the stem of Palm trees; stem 0·5-3·5 cm. high, simple or furcately branched, foliose, emitting rhizoids at their base, terete, without a central column; leaves recurved, up to 6-10 mm. long, sheathing and hyaline at their base, lanceolate to linear, often bearing brown brood-organs at their hyaline tip; true lamina occupying about $\frac{1}{4}$ of the width of the whole leaf on either side at and near their base, narrowing upwards and vanishing below the middle of the leaf, consisting of 5-10 rows of nearly square hyalocysts passing upwards into elongate-rhombic marginal cells; pseudolamina consisting near its base of 3-6, upwards of 8-10 layers of irregular, mostly hexagonal leucocysts; chlorocysts much narrower than the leucocysts, forming a single layer, rhombic and supra-median at the base of the leaf, higher up mostly triangular, arranged in a zigzag line and somewhat supra- or infra-median;

perichaetial leaves somewhat smaller than the stem-leaves; vaginula cylindric; seta thin, straight, up to 5 mm. long, when dry twisted; capsule prolate-spheroidal; peristome inserted below the orifice of the capsule and consisting of 8 simple short-lanceolate, yellowish teeth with a zigzag-shaped middle line; lid short-conical, ending in a curved point; calyptra hoodshaped reaching down to the middle of the urn; spores 12-25 μ in diameter, spherical, green, minutely warty.

The specimen described was collected in October, growing on the stem of Palm tree at Noakhali, Eastern Bengal.

Distribution :—Pantropical.

See—Engler's "Die natürlichen Pflanzenfamilien," second edition, Vol. X, pp. 225-226, and Fleischer's "Die Musci der Flora von Buitzenzorg," pp. 169-171.

Family: **Pottiaceae.**

Subfamily: **Trichostomoideae.**

Genus: **Hyophila.**

Species: **Hyophila involuta** (Hooker) Jaeger.

=*H. cylindrica* (Hooker) Jaeger.

Forming extensive, dense cushions on walls; stem simple, erect, emitting basal rhizoids, somewhat defoliated near the base, foliose upwards, with a central column; cells of the central column numerous, very narrow, of the fundamental tissue much wider and irregularly polygonal, peripheral cells narrow, rectangular in cross-section; leaves, when dry, incurved, obovate-subspatulate, 2-2½ mm. long and ½ mm. broad at the base and ¾ mm. broad at the apex, acute, margin denticulate towards the apex, when dry incurved, when moist flat; midrib strong, attenuated upwards, extending close to the tip of the leaf, with median deuter cells, forming in cross-section a single row associated with a dorsal and a ventral band consisting of numerous stereids covered by a dorsal and ventral layer of peripheral cells, rectangular and larger in cross-section than the stereids, and yellowish-green; lower leaf-cells hyaline, rectangular, considerably narrower near and at the margin, inner

cells 25-60 μ long and 6-8 μ broad; upper leaf-cells irregularly polygonal, 2-4 μ in diameter, with a papilla over the lumen, the upper marginal cells here and there denticulately protruding; hyaline part $\frac{1}{4}$ of the whole leaf; perichaetial leaves smaller than the lower leaves, the midrib reaching to about $\frac{1}{2}$ of the leaf, very delicate; seta erect, 8-10 mm. long, reddish-brown; capsule erect, prolate-spheroidal to subcylindric about 1·5 mm. long and 0·5 mm. in diameter, rim thick; exothecium consisting of rectangular, rather thick walled cells; neck short; lid large, conical, pointed, scarcely beaked; peristome absent, calyptra narrowly hoodshaped, twisted; spores 13-14 μ in diameter, translucent.

Found on the walls at Baliganj, Calcutta; mature capsule found in September and October.

Distribution.—Bengal, Nepal, W. Himalaya, South India, Tonkin.

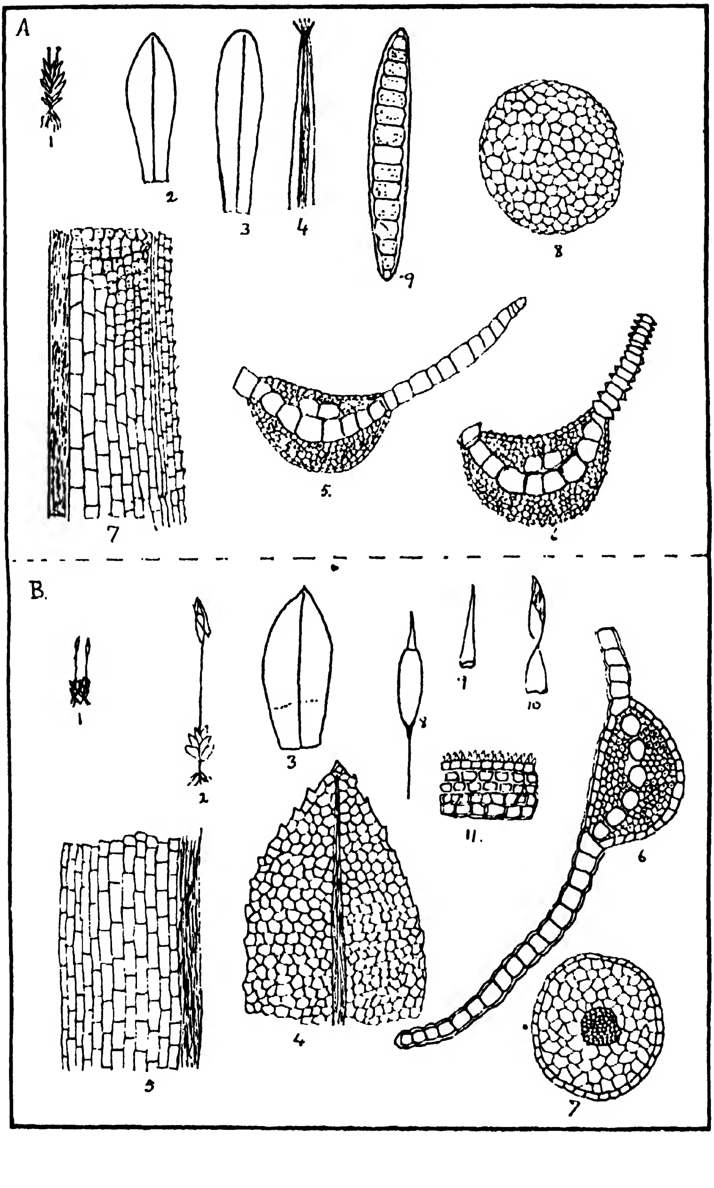
See—Engler's "Die naturlichen Pflanzenfamilien," second edition, Vol. X, p. 270.

Genus: **Barbula.**

Species: **Barbula indica** (Schwaegrichen) Bridel.

Plant-mass dense or somewhat lax, soft, yellowish-bright-green, less or more than 1 cm. in height; stem mostly simple, thin, nearly uniformly or somewhat laxly foliose, at the base with smooth rhizoids, in cross-section nearly circular, with a distinct axial column, the fundamental tissue-cells irregularly polygonal in cross-section, very thin walled, the peripheral cells coloured brown, narrower, rectangular in section, in one (-3) rows; leaves, when dry, variously curved or flexuose, moist erecto-patent, lanceolate, obtuse at the apex margins more or less recurved, minutely granulate due to the projecting papillae, midrib biconvex, ending in a small hyaline point slightly projecting beyond the leaf-blade, deuter-cells 4 (2-4) somewhat supramedian accompanied by a dorsal and a somewhat thicker ventral band of substereid-cells, peripheral cells much wider; lower leaf-cells hyaline or with a few chlorophyll grains, rectangular, 20-50 μ long and 4-6 μ broad, diminishing in width towards the margin, smooth; upper leaf-cells irregularly polygonal to nearly square, chlorophyllose, densely papillose both dorsally and ventrally about 4 μ in diameter; perichaetial leaves scarcely different, vaginula

PLATE II.



ovoid: seta erect, thin, reddish below, more yellowish upwards, finally entirely reddish; capsule erect prolate-spheroidal; neck very short; ring not differentiated; lid conical, ending in a short beak; peristome without a basal membrane, consisting of 32 filiform reddish, papillose, somewhat sinistrosely twisted segments; calyptra hoodshaped, scarcely reaching down to the middle of the urn; spores 7-10 μ in diameter; yellowish green, smooth.

Found growing on walls at Baliganj, Calcutta.

Distribution.—Bengal, Nepal, Ceylon, Tonkin, Malay Peninsula, Indian Archipelago, New Guinea.

See—Engler's "Die naturlichen Pflanzenfamilien," second edition, Vol. X, p. 279.

Family : **Entodontaceae.**

Genus : **Erythrodontium.**

Species : **Erythrodontium julaceum** (Hooker) Paris.

Plant-mass dense, extensive, yellowish-green to golden-brown, somewhat shining, covering the bark of the stems of trees; primary stem creeping, emitting here and there clusters of rhizoids, foliose or covered with the remains of old leaves, usually irregularly or one-sidedly pinnately branched, with a narrow axial column and the fundamental tissue consisting of thin-walled cells; branches subequally or unequally long, slightly bent or flexuose, most of them about 1 cm. long, catkin like and densely foliose; branch-leaves, when dry, adpressed, when moist, erecto-patent, ventrally very concave, broadly ovate-elliptical, contracted at the apex into a short broadish point, 1-1·5 mm. long and about 0·8 mm. broad; leaf-margin flat or slightly recurved, for most of its length entire, only at the point minutely serrate; midrib absent; leaf-cells smooth, the median basal and the upper prosenchymatic cells narrowly subrhombic-elliptic, often slightly bent, 50-80 μ long and 6-8 μ broad; alar cells arranged in oblique rows forming a broadly lanceolate, sharply defined alar area, broadly rhombic to irregularly polygonal or nearly square or transversely rectangular, about 10-20 μ long and 10-24 μ broad; perichaetial leaves small, narrow elliptic, shortly pointed, the inner larger and forming a sheath and ending in a serrulate point, without a rib; vaginula

cylindric; seta erect, yellowish-red, 1-2 cm. long, twisted; capsule erect, prolate-spheroidal, ring not differentiated, peristome double, inserted below the orifice, exostome consisting of 16 lanceolate, slightly remote reddish brown teeth, endostome of 16 short, filiform segments about ½ the length of the exostome-teeth; lid short, ending in a curved beak; calyptra hoodshaped, straw yellow, covering more than half of the urn; spores greenish, coarsely papillose, 18-30 μ in diameter.

Found on the bark of trees in the district of Mymensingh, Eastern Bengal.

Distribution.—Bengal, Assam, Khasia, Sikkim, Nepal, Nilgiris, Mysore, Burma, Yunnan, Tonkin, Ceylon, Indian Archipelago, Philippines.

See—Mitten, "Musci Indiae Orieantalis" in the Journal of the Linnean Society, 1859, p. 92 (*Stereodon juliformis*); and Fleischer's "Die Musci der Flora Von Buitenzorg," Vol. 4, pp. 1138-1141; Engler, "Die natürlichen Pflanzenfamilien, Vol. XI, p. 382.

EXPLANATION OF THE FIGURES.

(*Magnifications of the figures about time less than the given magnifications.*)

Plate I.

A. **Octoblepharum albidum** Hedwig.

Fig. 1. *Plant* $\frac{1}{1}$.
" 2. *Plant* $\frac{5}{1}$.
" 3. *Leaf* $\frac{7}{1}$.
" 4. *Cross-section of the upper part of the leaf* $\frac{82}{1}$.
" 5. *Cross-section of the basal part of the leaf* $\frac{82}{1}$.
" 6. *Cross-section of the stem* $\frac{109}{1}$.
" 7. *Capsule* $\frac{7}{1}$.
" 8. *Capsule with peristome* $\frac{10}{1}$.
" 9. *Calyptra* $\frac{16}{1}$.
" 10. *Lid* $\frac{8}{1}$.
" 11. *Peristome* $\frac{146}{1}$

B. **Leucophanes octoblepharoides** Bridel.

Fig. 1. *Plant* $\frac{2}{1}$.
" 2. *Side view of the leaf* $\frac{4}{1}$.
" 3. *Leaf flattened out* $\frac{7}{1}$.

PLATE III.

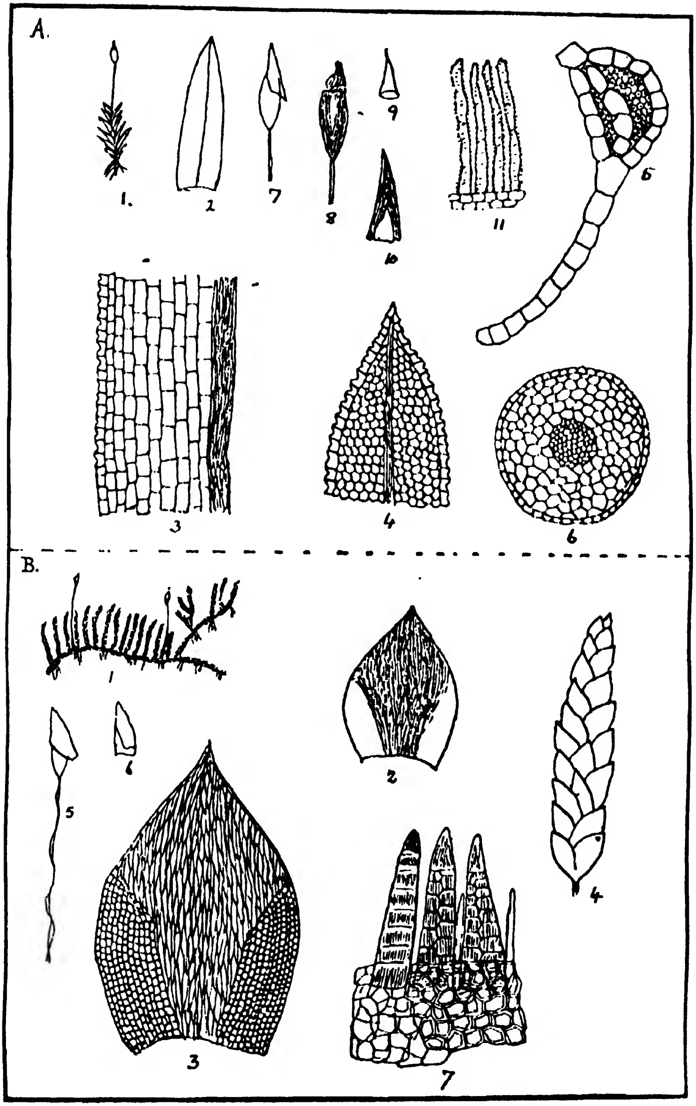


Fig. 4. *Upper part of the leaf* $\frac{16}{1}$.
„ 5. *Basal part of the leaf* $\frac{51}{1}$.
„ 6. *Cross-section of the middle part of the leaf* $\frac{82}{1}$.
„ 7. *Cross-section of the basal part of the leaf* $\frac{82}{1}$.

Plate II.

A. **Calymperes noakhalensis** Brühl et Sarkar.

Fig. 1. *Plate* $\frac{1}{1}$.
„ 2. *One of the basal leaves* $\frac{13}{1}$.
„ 3. *One of the middle leaves* $\frac{12}{1}$.
„ 4. *Abnormal leaf bearing brood-organs* $\frac{82}{1}$.
„ 5. *Cross-section of the basal part of the leaf* $\frac{82}{1}$.
„ 6. *Cross-section through the upper part of a leaf, margins and tenioles not shown* $\frac{82}{1}$.
„ 7. *Basal part of a leaf* $\frac{75}{1}$.
„ 8. *Cross-section of the stem* $\frac{82}{1}$.

B. **Hyophila involuta** (Hooker) Jaeger.

Fig. 1. *Plant* $\frac{1}{1}$.
„ 2. *The same* $\frac{2}{1}$.
„ 3. *One of the leaves* $\frac{12}{1}$.
„ 4. *Upper part of the leaf* $\frac{41}{1}$.
„ 5. *Basal part of the leaf* $\frac{41}{1}$.
„ 6. *Cross-section of the leaf* $\frac{82}{1}$.
„ 7. *Cross-section of the stem* $\frac{82}{1}$.
„ 8. *Capsule* $\frac{10}{1}$.
„ 9. *Lid* $\frac{15}{1}$.
„ 10. *Calyptra* $\frac{15}{1}$.
„ 11. *Part of the mouth of the capsule* $\frac{62}{1}$.

Plate III.

A. **Barbula indica** (Schwaegrichen) Bridel.

Fig. 1. *Plant* $\frac{3}{1}$.
„ 2. *Leaf* $\frac{18}{1}$.
„ 3. *Basal part of the leaf* $\frac{62}{1}$.
„ 4. *Upper part of the leaf* $\frac{62}{1}$.

Fig. 5. *Cross-section of the leaf* $\frac{82}{1}$.
„ 6. *Cross-section of the stem* $\frac{82}{1}$.
„ 7. *Capsule* $\frac{10}{1}$.
„ 8. *Capsule with twisted peristome* $\frac{15}{1}$.
„ 9. *Lid* $\frac{15}{1}$.
„ 10. *Calyptra* $\frac{18}{1}$.
„ 11. *Part of the peristome, teeth separated* $\frac{82}{1}$.

B. **Erythrodontium julaceum** (Hooker) Paris.

Fig. 1. *Plant* $\frac{1}{1}$.
„ 2. *Leaf* $\frac{32}{1}$.
„ 3. *Leaf more highly magnified* $\frac{82}{1}$.
„ 4. *A single branch* $\frac{20}{2}$.
„ 2. *Capsule* $\frac{10}{1}$.
„ 6. *Calyptra* $\frac{15}{1}$.
„ 7. *Peristome* $\frac{82}{1}$.

The Development of the Embryo-Sac in Carica Papaya.

By

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Ghose Professor of Botany,

and

Ilabonto Banerji, M.Sc.,

Ghose Research Scholar in Botany.

(Completed, December 20th, 1928.)

The study of the development of the female gametophyte in angiosperms has received considerable attention from investigators in recent years. **Coulter** and **Chamberlain** (1) have summarised the work done up to 1903; while **Rutgers** (4) gives an account of the work done up to 1923. The recent publications of **Schürhoff** (6) and **Schnarf** (5) bring the information on the subject up to date.

Although cultivated very widely in the tropics the gametophytic generation of *Carica Papaya* has not been studied in detail. **Siguara** (7) and also **Sutaria** and **Asana** (8) studied the microsporogenesis only.

Usteri (9) in 1903 gave the first account of the embryo-sac development in *Carica Papaya*. He observes that a normal linear tetrad is formed of which the first cell functions as the megaspore. The antipodals are seldom developed. **Kratzer** (3) working on the same subject in 1918 states that three to four "sporenzellen" are developed of which any one may develop into the megaspore. The development of the embryo-sac is normal. **Heilborn** (2) in 1921 working on the gametophytic development of the various species of *Carica* from Equador obtained entirely different results from those of Usteri and Kratzer. His investigations led him to the conclusion

that the "megaspore mother cell divides, but no cell-wall and consequently no tetrad is formed and the mothercell gives rise to a binucleate embryo-sac." Further division of these two nucleii produces a quadrinucleate embryo-sac. Of these only one divides and a pentanucleate embryo-sac results. No further division of the nucleii takes place. The mature embryo-sac, therefore, contains five nucleii of which three form the egg apparatus, and the other two the polar nucleii, the antipodals being absent.

Such divergence of views made a re-examination of the subject necessary and this we have attempted here.

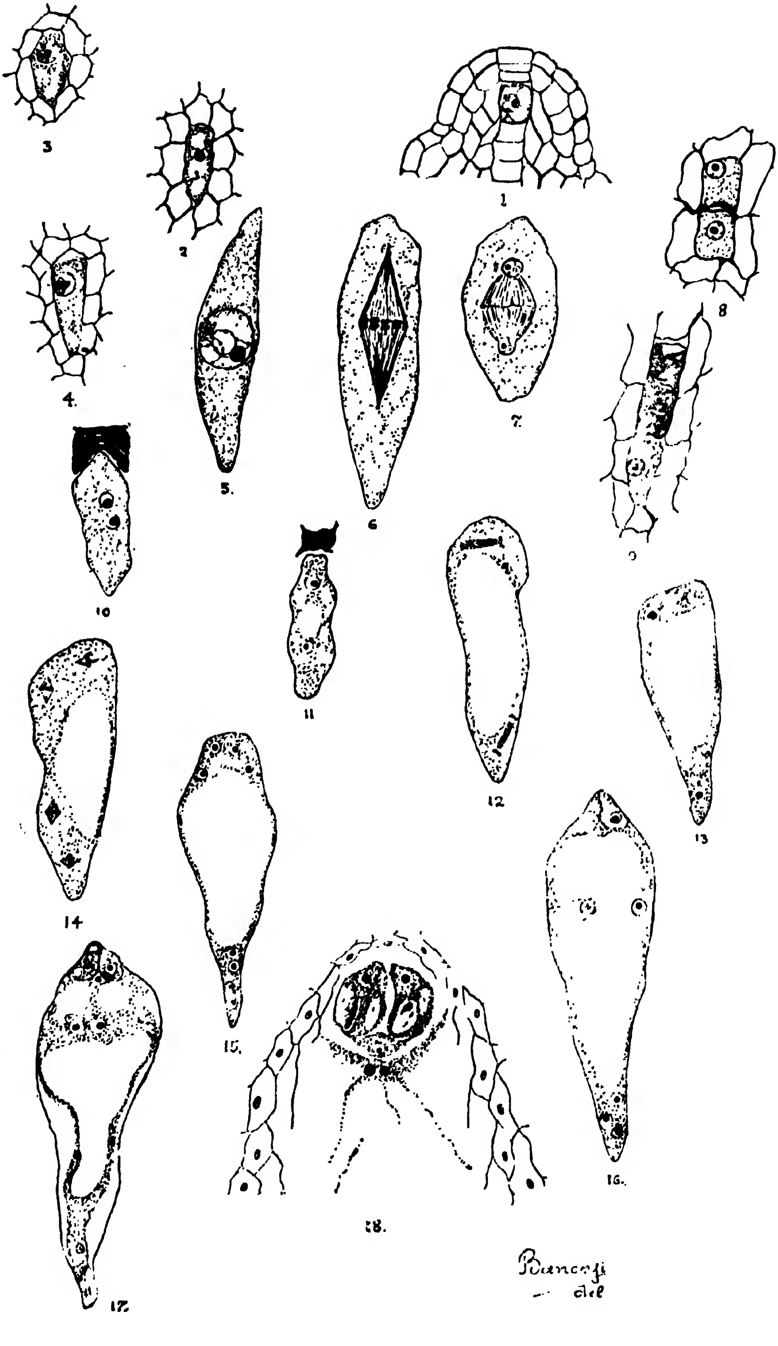
Material and Methods.

The material used in this investigation was collected from two plants growing in the University College gardens. Of these one was diœcious and other monœcious. Material from these two plants was always collected and kept separate. Collections of material were usually made on bright days between 12 noon and 4 P.M. Buds in all stages of development were fixed, as well as fully opened flowers. The latter were always collected and fixed separately. To facilitate penetration by the fixing fluid the ovaries were trimmed and in certain cases cut longitudinally before fixation Two fluids were first tried, Flemming's weak fluid and Allen's modified Bouin's fluid. The latter showed very good fixation and as such was exclusively used later. The fixing fluid was heated to an initial temperature of 40°C before fixation which was always done in the field. The material was allowed to remain in the fixing fluid for a period varying from twelve to twenty hours depending on the size of the material. After fixation the material was run up to 70% alcohol in the course of an hour and repeatedly washed in 70 per cent. alcohol every twelve hours till all the green colouration disappeared. It was then dehydrated, cleared, embedded and sections were cut 6, 8, & 10μ in thickness. Heidenhain's iron-alum hæmatoxylin was used for staining.

Macrospore Development.

The ovules of *Carica Papaya* are very numerous in number and are not restricted to the placentas but sometimes develop from the

Plate I



entire inner-wall of the ovary. They are anatropous, and possess two integuments and a many-layered nucellus.

The macrospore mother cell in the young stage could be distinguished from the surrounding cells of the nucellus by its comparatively larger size, its conspicuous nucleus and the characteristically vacuolated condition of its protoplasm. The mother cell appears in the hypodermal layer of the tip of the nucellus, and is pushed inwards by 6-8 layers of covering cells. In no case was any ovule found to contain more than one mother cell. The archesporial mother cell gradually increases in size being somewhat broad. at the micropylar end and narrow towards the chalazal (Fig. 2). The nucleus then passes into the synaptic stage and throws out characteristic loops. The position of the nucleolus at this time is more or less central (Figs. 3 and 19) but it gradually moves and takes a peripheral position (Figs. 4 and 20.) The spireme appears to be beaded in nature (Fig. 5). The heterotypic division was observed in many preparations (Fig. 6). It was not possible to count the numbers of chromosomes but their dyad nature could be made out when on the equatorial plate. In Figure 7 a late telophase stage is represented together with the initial stages in the formation of the cell plate. Figure 8 represents a later stage when the division of the cell is complete. No typical and clear homeotypic division was observed but that it does occur is proved by the abundant occurrence of linear tetrads, generally with the first three cells in the disintegrating stage (Fig. 9).

It should be mentioned here that in very young ovaries different stages of macrospore development were observed. In some ovules the archesporial mother cell was found in the resting stage, whereas in others it had completed the tetrad division.

Embryo-sac Development.

The innermost or the chalazal spore-cell becomes the embryo-sac while the others rapidly disintegrate (Fig. 9). The disintegrating cells stain very deeply and appear as dark shapeless masses even before the initiation of the first division of the nucleus of the chalazal megaspore. Their presence could be detected even up to the second division of the embryo-sac nucleii (Fig. 10).

The embryo-sac increases in size before division. The nucleus lies in the centre and at this stage no vacuolation of the surroundign

cytoplasm is noticeable (Fig. 9). After the first division has taken place the daughter nucleii move away from one another, one to each end of the embryo-sac and the beginnings of a central vacuole are noticed (Fig. 11). The embryo-sac approximately doubles itself in size before the division of the daughter nucleii and the central vacuole increases in size. The micropylar end of the embryo-sac is rather broad at this stage, while the chalazal end is somewhat pointed (Fig. 12). This characteristic appearance is maintained up to the octo-nucleate stage. In Figure 12 the two daughter nucleii are also seen undergoing division. In Figure 13 a quadrinucleate stage of the embryo-sac is represented. The vacuole, it will be noted, has become larger and the daughter nucleii are exactly similar in appearance. In Figure 14, a quadrinucleate stage of the embryo-sac is seen in the metaphase stage. This leads to the octo-nucleate stage which is represented in Figure 15. It will be noted that from the bi-nucleate stage onwards the nucleii are distributed equally at both the chalazal and micropylar ends of the embryo-sac. In the octo-nucleate stage the embryo-sac has increased very much in size and disintegration of the adjacent cells of the nucellus has commenced. The central vacuole has increased largely in size and has only a delicate lining of cytoplasm at the sides. The nucleii are distributed equally four at the chalazal and four at the micropylar end of the embryo-sac. Two nucleii, one each from the chalazal and the micropylar end of the embryo-sac, then migrate towards the centre and form the polar nucleii (Fig. 16). They do not fuse but lie very close together as shown in Figure 17. The antipodals as well as the central vacuole become disorganised at this stage, and no trace of them is noted in the subsequent differentiation of the embryo-sac. The polar nucleii then move upwards and lie very close to the egg-apparatus as depicted in Figure 18. The fully mature embryo-sac thus contains the two synergids, the egg-cell and the two polar nucleii. The polar nucleii which are represented in Figure 18, remain surrounded by the cytoplasm just below the egg cell, and in very close contact, but probably do not fuse till the appearance of the second male gamete. The distinction between the synergids and the egg-cell is quite marked at this stage. The egg-cell is situated centrally between the synergids and the polar nucleii, its nucleus is comparatively small in size and the cytoplasm is scanty. The synergids are characterised by their comparatively larger size, the presence of a

number of vacuoles, and their comparatively bigger nucleus. A fully differentiated embryo-sac is represented in Figure 18, in which the structure and orientation of the component parts are clearly depicted.

The fully opened flowers which were generally fixed two to three hours after anthesis showed embryo-sacs in all stages of development. Fully differentiated embryo-sacs as well as embryo-sacs in a state of disorganisation (Fig. 21) were also noted, the proportion of the latter however, being very small.

Discussion.

The very variable results obtained by previous workers on the subject led us to believe at the inception of our investigation that these diverse results might be accounted for by the material used having been different. *Carica Papaya*, as is well known, is a diœcious plant, although certain forms are known to be monœcious. Bisexual flowers have also been occasionally noted. Material from monœcious and diœcious plants was fixed and worked out separately with the idea that the origin and development of the embryo-sac might show some differences. Critical study of the various stages, however, has failed to reveal any differences. The detailed structure of the egg, the synergids, and the polar nucleii is identical in both the monœcious and diœcious forms. In both, the disintegration of the antipodals takes place immediately after the octo-nucleate stage has been reached. In fact no difference in the origin and development of the embryo-sac was noted in the two forms and as such it has not been mentioned in the text.

The question arises how then is it possible to explain the diverse views on the subject ? Our conclusions differ in certain fundamental points with those of Usteri (9) and Kratzer (3). According to Usteri (9) the first cell of the tetrad functions as the megaspore mother cell, while Kratzer (3) states any one cell of the tetrad might function as such. Critical examination of a large number of preparations of both the forms by us clearly shows that the last or the chalazal cell of the tetrad always develops into the megaspore while the others abort. Our results also differ from those of the above-mentioned investigators as regards the development of the antipodals. Usteri (9) observes that antipodals are seldom developed while Kratzer (3) states

that the development of the embryo-sac is normal. As the antipodals are developed in the tail-like projection of the embryo-sac (*vide* Fig. 15) and are transitional in character, Usteri's (9) statement may possibly be due to his not having examined a sufficiently large number of stages. Kratzer (3) never mentions the disintegration of the antipodals, nor gives sufficient figures from which one could trace the course of development. Heilborn's (2), investigations differ fundamentally from those of Usteri's (9), Kratzer's (3) and ours. It is not possible to account for his results except on the supposition that the Equadorian species of *Carica* differ from the rest in their gametophytic development.

Summary.

1. The archesporial cell has its initiation in the hypodermal layer of the nucellus.

2. A normal linear tetrad is formed of which the first three cells disintegrate and the last functions as the megaspore.

3. The megaspore nucleus divides in the usual manner and produces an octonucleate embryo-sac.

4. The antipodals are ephemeral in nature and become disorganised before the differentiation of the egg-apparatus.

5. The fully differentiated embryo-sac contains the egg-cell, the two synergids, and the two polar nucleii.

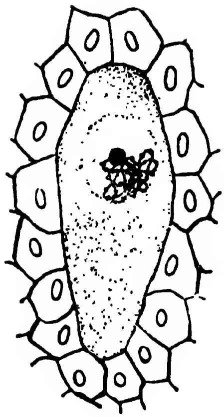
6. The polar nucleii lie close together but do not fuse.

7. The development of the embryo-sac is the same in both the monœcious and diœcious forms of *Carica Papaya*.

Literature.

1. Coulter and Chamberlain ... Morphology of Angiosperms (1903).
2. Heilborn, O. ... Taxonomic and cytological studies on cultivated and Ecuadorian species of Carica. Arkriv. för Bot. 17. Nr. 12 (1921-22).
3. Kratzer, J. ... Die verwandtschaftlichen Bezi ehungen der Cucurbitacean auf Grund ihrer Samen-entwicklung. Flora Bd. 110. (1918).

Plate II



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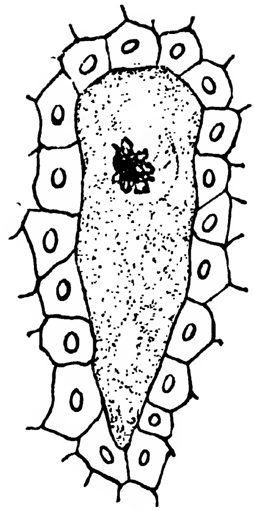

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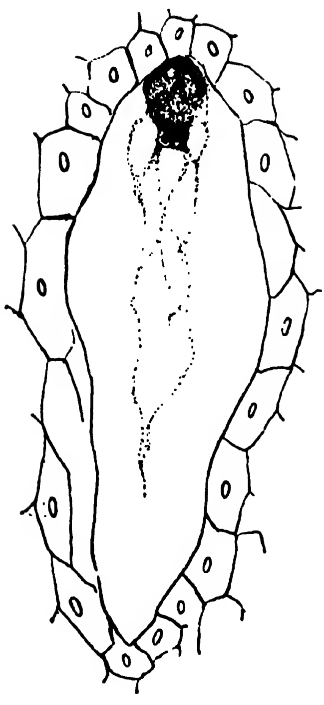

21.

4. Rütgers, F. L. ... The female gametophyte of angiosperms. Ann. Jard. Bot. Buitenzorg. 33 : 6-66 (1923).

5. Schnarf, K. ... Embryologie der Angiospermen (1928).

6. Schürhoff, P. ... Cytologie d. Blütenpflanzen (1926).

7. Siguara .. Some observations on the meiosis of pollen mother cells of *Carica Papaya*, *Myrica rubra*, *Aucuba japonica* and *Beta vulgaris*. Bot. Mag. Tokyo. 41: 219-224 (1927).

8. Sutaria, R. N. and Asana J. J. A cytological study of pollen development in *Carica Papaya* (from Proceedings 15th Ind. Sc. Congress, Cal., 1928).

9. Usteri, A. ... Studien uber *Carica Papaya* L., Ber. d. deutsch. bot. Ges. Bd. XXV (1907).

Explanation of Plates.

All figures were drawn with the aid of camera lucida at an initial magnification of × 540, with the exception of Figures 5, 6, 8 and 19, 20 and 21 which were drawn at a magnification of × 960 and 1450 respectively.

1. Longitudinal section of ovule with differentiated archesporial cell which has already been pushed inwards by layers of covering cells.

2. Megaspore mother cell beginning to enlarge.

3. Synaptic stage of megaspore mother cell with nucleolus in the centre.

4. Synaptic stage of megaspore mother cell with nucleolus at the side.

5. Synaptic stage of nucleus ; note beaded nature of spireme.

6. Heterotypic metaphase stage.

7. Late telophase stage.

8. Completion of division.

9. Linear tetrad with the first three cells in disintegrating stage.

10. Binucleate stage of Embryo-sac.

11. Binucleate stage of Embryo-sac ; showing the initiation of central vacuole.

12. Binucleate stage of Embryo-sac ; daughter nucleii at telophase stage.

13. Quadrinucleate stage of Embryo-sac. Note central vacuole (reconstructed from two successive sections).

14. Quadrinucleate stage of Embryo-sac, daughter nucleii in metaphase stage.

15. Octonucleate stage of Embryo-sac. Note tail-like projection of chalazal end.

16. Octonucleate stage of Embryo-sac showing the migration of polar nucleii (reconstructed from two successive sections).

17. Octonucleate stage of Embryo-sac showing disintegration of central vacuole and antipodals.

18. Fully developed Embryo-sac.

19. Synaptic stage of megaspore mother cell with nucleus in centre.

20. Synaptic stage of megaspore mother cell with nucleolus at the side.

21. Fully developed Embryo-sac in process of degeneration.

On the method of multiplication of Pentatrichomonas and Trichomonas and the origin and development of their organelles.

By

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With Plates, I-III.

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Introduction.

Dobell and O'Connor (1921), writing about the method of reproduction in *Trichomonas* in their treatise on Intestinal Protozoa of Man, states, " *Trichomonas hominis* multiplies in the bowels by longitudinal division, but stages in the process are excessively rare in the stools, and consequently no account of it can yet be given. The division of *Trichomonas* is a complicated process and conflicting accounts of the details as observed in other species have been published."

Kofoid and Sweezy (1915) state that the daughter *Trichomonas* formed as a result of multiple division, which they describe, resemble even in their early stages the parental form regarding the arrangement of flagella, axostyle and the undulating membrane. No other observer has yet confirmed this mode of reproduction.

Chatton (1918 *b*) in culture of *Trichomonas caviae* found some forms in which all the flagella (4 in number) are free, none being attached to form the undulating membrane.

But observers like Wenyon, Hartmann, Dobell, Reuling and others agree in believing that the only method of reproduction of *Trichomonas* is by binary fission, by which the elements formed after division possess, besides the three flagella, an attached flagellum forming the border of the undulating membrane. It is very difficult to reconcile the observation of Chatton with the only accepted method of reproduction in *Trichomonas*, namely, by binary mitosis.

Wenyon notes that in the method of division by binary mitosis, axostyle divides into two and persists throughout the division, while Wenrich, Tanabe and others state that in the process of binary division, the axostyle disappears and new ones are formed from the paradesmose.

Lastly, Reuling, Kofoid, Hartmann, Alexief and others believe that the axostyle of *Trichomonas* is of flagellar origin and bears a homology with the axostyles of *Hexamitus* and *Giardia*. Wenyon, however, is of opinion that the axostyle of *Trichomonas* is not of flagellar origin, it is of nonstainable skeletal structure and so not homologous with the axostyle of *Hexamitus* and *Giardia*, which, according to him, ought not to be called axostyle at all.

All these conflicting opinions point to the necessity of the problem being further investigated.

As we in studying cultures of various species of *Trichomonas*, observed some phenomena, which can explain away a good portion of these conflicting views, we thought it proper to publish our observations on the subject. It is necessary to state at the outset that one of us has already published a preliminary note on the same topic; ours is not only a confirmation of it, but a more exhaustive dealing with the same and recording of some new facts not observed before. Though we found, nearly the same phenomenon in cultures of all the varieties of *Trichomonas* observed hitherto by us including the human one (*Pentatrichomonas*), we will deal mainly with *Pentatrichomonas bengalensis* (Chatterjee, 1915) and *Trichomonas mabuia* (Dobell, 1910) in this paper, and incidentally we will refer to *Trichomonas caviae*.

Method and Technique.

The organisms were cultivated at room temperature, in a medium composed of normal saline mixed with a little laked blood. The culture was made in a glass tubing, one end of which was drawn to a fine point and sealed. This pointed end was the lower portion, the upper cut end being plugged with cotton. For examination we drew out the deposit from the bottom of this tube, by means of a capillary pipette. In the method of making slide preparations, we used streak method, *i.e.*, in emptying the contents of the capillary tube, a series of lines were made instead of making a smear on the slide. To make a successful streak a perfectly fat-free slide is necessary. We followed

this method both when we stained the preparation by Iron Haematoxylin after wet fixation or by modified Leishman after dry fixation. By the former procedure we were able to get the greatest number of organisms to be found in the specimen, because multiplication of *Trichomonas* takes place at the bottom, very few being found on the surface or near the surface. By the latter procedure, namely, by the streak method of making slide preparation, we were enabled to bring to the field of vision any and every portion of the preparation, no portion escaping observation. Besides, numbering the streaks and putting down the place in the streak, corresponding to the vernier reading in the mechanical stage in our note book, we can easily find out the particular organisms, exhibiting characters worth noting. To this method of making preparations, we owe the finding of the characters of the numerous organisms found in the culture which escaped so long detection by other observers in this line. In preparing the Leishman stain used in this work, chromatic element in the "Nocht Rot" has been developed as much as possible by prolonged exposure to tropical sun and at the time of drawing the streak the material was diluted with fresh serum to rejuvenate it. We were able to get also, by using this method, the beautiful staining of the structures of the *Trichomonas* in their minutest details in the fully developed organisms 18 μ in length, as well as the smallest gemmules measuring not more than 2 μ, in which all the organelles were also seen clearly; besides we got differential staining (chromatic lines stained red, some other structures bluish red and others again intermediate stains),—a thing not obtainable in the method of staining by Iron Haematoxylin, whatever method of fixation be followed (Bouin, Schaudin or Flemming). Besides, in Iron Haematoxylin method, due to shrinkage during fixation, observation of the interior of the minute gemmules is not possible. The value of this extra developed chromatic stain in our Leishman stain, is proved by the staining of the peritrichous and the end flagella of bacilli present in the film (this is not possible in ordinary Leishman stain even if we stain it overnight). A paper on the subject of flagella of bacilli being of the same nature as those of Protozoa, as is evidenced by both being stained by the same method, will be published by us later on.

As a result of our observation, we can assert that we have succeeded in finding in *Pentatrichomonas* and *Trichomonas*, a method of multiplication which has not been observed before. The mode of

multiplication found by us resembles more than any thing else the method described by Kofoid and Sweezy (1915) in connection with *Trichomonas* under the name of Somatella formation or multiple fission. Though our observation is not exactly a confirmation of the finding of these authors, differing, as it does, in several material respects, yet it will be best to designate it, Multiple fission by Somatella formation.

Examination of the Specimens.

Now, from the study of the stained smears of a rich culture of *Pentatrichomonas* (either directly from stool or from culture) or of culture of *Trichomonas*, we can trace all the developmental stages from non-flagellate plasmodial forms measuring not more than 2 to 3 μ (Figs. 35 and 37) to fully developed *Pentatrichomonas* forms measuring 18 μ in length and 8 μ in breadth (Fig. 1). Before we proceed to describe them and their mode of origin, it is necessary to mention that we took steps to exclude the possibility of our culture from being contaminated by any coprozoic flagellate growing in the culture and thereby vitiating our observation. The finding only of bodies whose structures show that they are some developmental forms of a *Trichomonas* and the absence of any other protozoal organism, in all our cultures, show that no such contamination took place. We can adduce further that we several times succeeded in growing our culture from a single *Pentatrichomonas* obtained from human stool seen under low power, in a hanging drop culture, in which we were sure that there was no other protozoal organism. Besides, we succeeded in making a culture from the fluid contained in a capillary tubing which was examined by low power which showed only a single *Trichomonas*.

The organism found in the culture as seen in stained film, can be classified under the following heads :—

A. Several developmental stages from plasmodial forms to fully developmental forms.

B. Binary divisional forms.

C. Multinucleated somatella forms.

D. Multinucleated somatella forms with gemmules.

Before proceeding further, it is necessary to state our idea, how multiplication takes place in *Trichomonas*. Presence of numerous

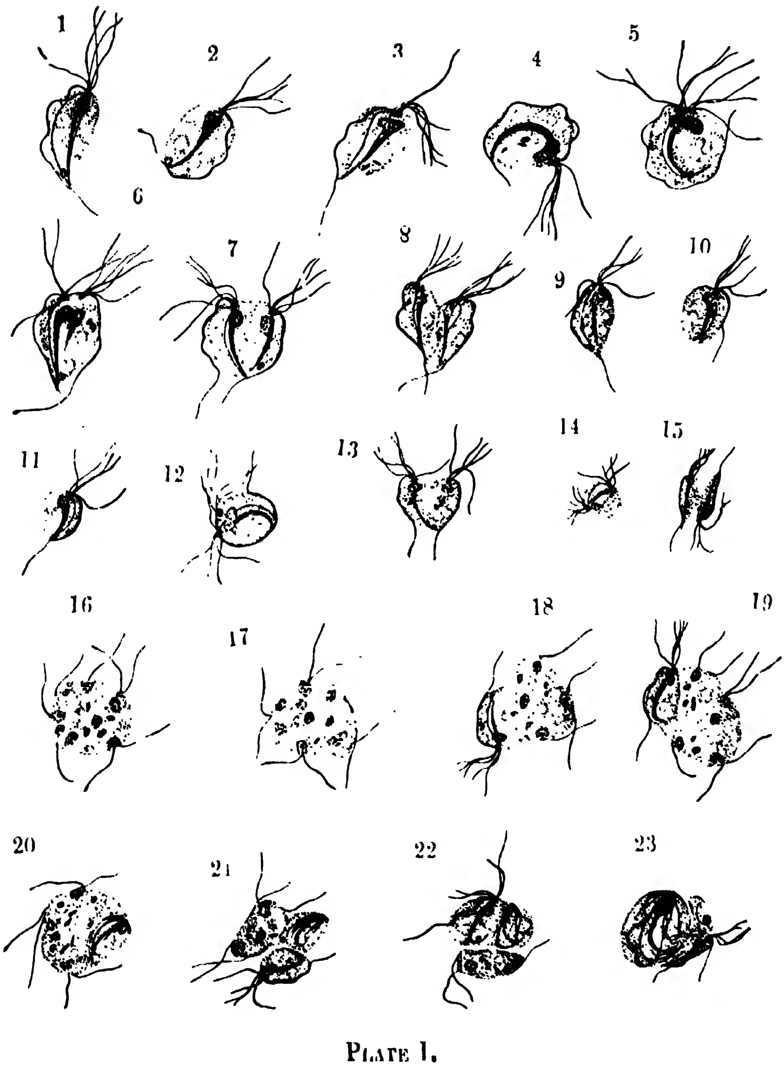


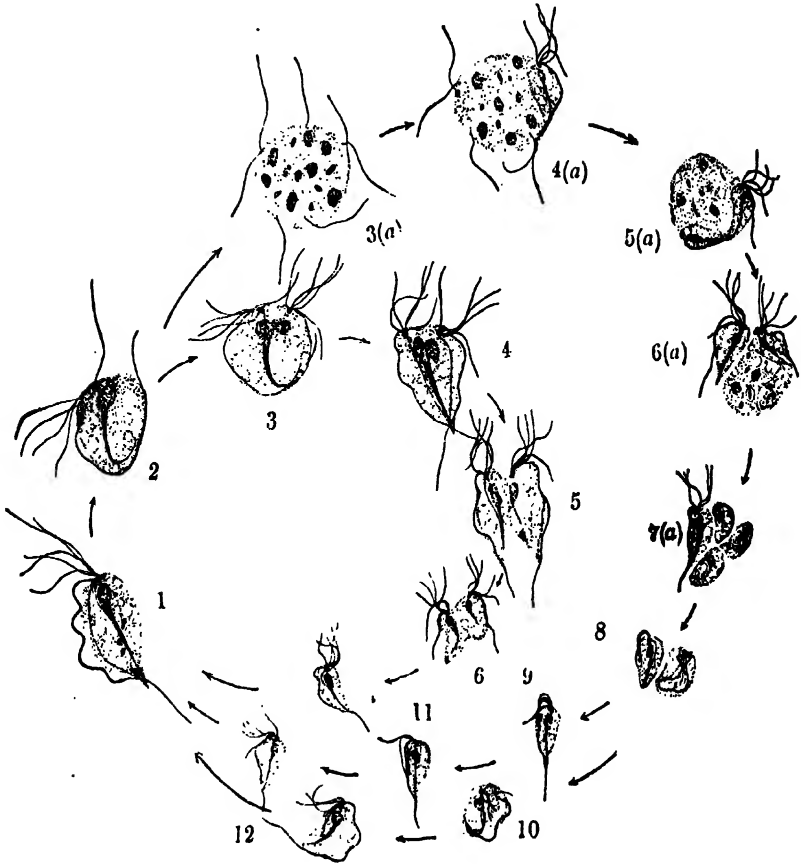
PLATE I.

small plasmodial forms as well as small developmental forms, showing flagella but no undulating membrane points to these originating by some other method of multiplication than by mitosis which is the only method noted by observers like Wenrich, Wenyon and others. This also excludes the possibility of the forms originating from somatella forms as described by Kofoid, and Sweezy, in which each unit after division shows a fully developed undulating membrane and three free flagella.

Our idea about multiplication of *Pentatrichomonas* and *Trichomonas* is as follows :—

The study of the forms like those seen in Pl. I, Figs. 16, 17 and 18, showing multinucleated flagellated forms will give a clue to the method of multiplication and so we will describe them first. The flagella of these multinucleated forms (somatella) are quite different from the flagella of the developmental forms of *Trichomonas* in being much finer and fainter. Besides, they arise singly from all sides of the body of the somatella. In the interior of the somatella are found several nuclei. These multinucleated flagellated forms differ from the somatella forms of Kofoid and Sweezy, in this respect, that the flagella of the somatella according to these observers furnish the compliment of flagella to the units of *Trichomonas* originating from the somatella, whereas in our case, these flagella do not—these flagella being evidently temporary structures, disappear after the gemmules are separated. These somatellas vary very much in size, the largest being 18 μ in diameter, smallest being 6 μ. They show as many as 8 nuclei and also several chromatic dots (basal granules=centrioles) and also several fine rods and threads (axonemes of flagella). Besides, a portion of some of the somatella is seen occupied by what appears to be a developmental form of *Trichomonas* (Figs. 18 and 19) showing a nucleus, basal granules, five free flagella (in case of *Pentatrichomonas*) and an attached flagellum and rudimentary axostyle. The flagella of this developmental form, which can be termed a gemmule, differ from the temporary flagella of the somatella, as stated before, in being much thicker and in taking more intense stain. They are definite in number and originate from a group of basal granules similar to the arrangement found in adult *Trichomonas*. It can be demonstrated that this gemmule separates out from the parent somatella after a time and becomes a motile organism and such gemmules are the source of the numerous free developmental forms of *Trichomonas* seen in the

culture. This can be made out clearly from the diagrammatic representation of the method of multiplication given in the text figure.



Diagrammatic Representation of Method of Multiplication of Pentatrichomonas bengalensis.

1, Adult *Pentatrichomonas* ; 2, rounded form of the same ; 3, 4, 5 and 6, ordinary method of multiplication by binary fission ; 7, an individual after division ; 3*a*, multiple division (somatella) ; 4*a*, a somatella with a gemmule ; 5*a*, a somatella with two gemmules, one in plasmodial stage and the other in flagellated stage ; 6*a*, separation of the gemmules ; 7*a* and 8, further stages in separation of gemmules ; 9, 10, 11 and 12, separated individuals from the somatella developing into adult forms.

After the separation of the gemmule, another gemmule is formed in connection with another nucleus including a portion of the cytoplasm of the somatella. It may so happen that two or three gemmules may form simultaneously, and these separate after a time in their turn. The process is repeated, till no more nucleus is available to give rise to a gemmule. The origin of the plasmodial forms also can be traced from these somatellas. The gemmule in this case does not develop enough to form flagella before separation (Fig. 27)—they do so after separation. Very often a flagellated gemmule may include only the nucleus, very little of the cytoplasm of the somatella being included in it, so that the appearances seen in Figs. 44 and 65 are found showing only a thick chromatic mass (nucleus) from which are seen originating flagella as well as axostyle, there being very little or no cytoplasm surrounding the nucleus. In case of *Trichomonas caviae*, we found that these gemmules are so small as to measure 1 to 2 μ in length and ·3 μ in breadth, in these the chromatic substance (nucleus) is found distributed throughout the body of the rod-shaped organism and not forming a definite nucleus, gives it the appearance of a bacillus ; from the possession of three anteriorly directed flagella and one as a trailing flagellum, it can be made out as a *Trichomonas* and not a bacillus. Besides, the flagellated free gemmule after separation from a somatella often divides by mitosis and gives rise to two flagellated organisms. Lastly, it is necessary to add that a large number of free gemmules are seen in which the flagella are seen occupying the cytoplasm of the gemmule and have not become free (Figs. 32, 74 and 75). These seem to be the origin of cyst about which a separate paper has been written. It can be made out that the organisms, originating either as gemmules from somatella or as a result of mitosis of a free one, gradually pass through several developmental stages before they develop into adult *Trichomonas*. All these developmental stages are seen in the culture in Plates II and III.

General Consideration.

Before we proceed to put our interpretation on the findings described above, it is necessary to state the views of some of the observers on the structure of the nucleus of flagellates in general and associated organelles and the relation of the organelles to each other.

It is needless to say that there is bound to be a good amount of divergent views on such abstruse matter as this. As Doflein in his studies of *Chrysomonadina* has gone very deeply into the subject and has reviewed the views of other workers in the field, it is best, in order to get a clear idea of the subject, to put his views first.

In studying the division of this flagellate which possesses, besides a chromatophore, two flagella originating from basal granules (Basalkörner of Doflein, Hartmann, etc., and blepharoplast of Wenyon, Dobell and Alexieff and others), Doflein states that "Basalkörner" has got structure and relation analogous to the centriole. Besides, the "Basalkörner" resembles centriole and centrosome in their power of dividing by themselves. They also resemble in this property of autonomy with chromosome, blepharoplasts (=Kineto nucleus of Mesnil, Kinetoplast of Wenyon) and chromatophore, etc. The "Basalkörnern" are differentiated from the chromatophore by the fact, that they can disappear completely and can reappear again.

In studying the relation of the "Basalkörner" to the poles of the spindle formed during division of the Protozoa, he comes to the conclusion that the "Basalkörnern" along with the flagella divide independently of the nucleus, long before any change can be observed in the nuclear structure. But during the telophase stage of nuclear division, the "Basalkörnern" with their flagella occupy the position of apices of the spindle. From this, he opines that the "Basalkörnern" are the source of energy and so correspond to centriole and centrosome. To decide whether they are actually centriole and centrosome or not, he draws certain conclusions from the study of Heliozoa. Here the axile thread which corresponds to a flagellum, originates from central korn which in *Actinophrys sol*, is situated in the nuclear membrane, whereas in *Actinosphaerium eichhorni*, it originates from plasma of protoplasm. Here also the Central korn has its origin independent of the nucleus and is the centre of energy of the Heliozoa, as from it streams of protoplasm originate at the time of movement. In division of the nucleus of the Heliozoa and also of the *Ochromonas*, structures situated at the pole of the spindle correspond to the "central korns," as centrioles do in the case of the spindle of the dividing metazoa cells.

Whether these centrioles of the spindle and the "Basalkörner" are one and the same thing, he does not decide. To decide

this point, he says that one must have a clear idea of what a centriole is.

Wenyon states in his treatise on Protozoa, that there is reason to believe that the blepharoplast (=basalkorn of Doflein) from which the axis (axoneme) of flagellum takes its origin, is of nuclear origin. Jameson states when a flagellum is about to be formed, a granule separates from the karyosome of the nucleus and passes out into the cytoplasm through the nuclear membrane and this serves as a basal granule for the flagellum; Wenyon states, " It is claimed that as the centriole is functionally a centrosome, the blepharoplasts of flagellates must also be centrosomes. It is further assumed that in those cases in which the blepharoplast occupies a position in the cytoplasm apart from the nuclei, it represents a centriole or centrosome into two parts one of which remains in the nucleus and still functions as a centrosome during its division, while the other has left the nucleus to become a blepharoplast." He does not however agree with the view that the blepharoplast originates from the karyosome, but remains somewhere in the cytoplasm near the nuclear membrane.

He says besides, " The whole subject of the relation of blepharoplasts to centrosomes is a very complex one, and depends largely on the exact definition of a centrosome. Some observers definitely assert that the blepharoplast is a centrosome."

Doflein in speaking of the basal korn asks, "Aber aus was besteht diese Substanz? Ist sie lebend oder tot? Ist sie Plasma selbst oder ein Produkt des Protoplasmas? Wir stehen da vor ungelösten Grundfragen. Dass die Basalkörner wachsen und sich teilen, gehört zu jenen Vorgängen, die wir als besonders charakteristisch für das Lebendige betrachten. Aber offenbar besitzen sie nicht die Kontinuität, welche für das Lebendige das wichtigste Kennzeichen ist. Sie können verschwinden, sich auflösen z. B. bei der Cystenbildung; sie können wieder neu entstehen. Darin gleichen sie vollkommen den Geisseln selbst."

Wilson in his treatise on " Cytology of Cell" states that centrosome of a cell whether of metazoa or protozoa, is found under varying circumstances in any part of the cell in the interior of the nucleus or near the nuclear membrane or anywhere in the cytoplasm and in many the centrosome is not a single body, but several centrosomes can be found in the same cell. Its position in the cell is an accidental factor and not connected in any way with the nucleus.

Our Interpretation of the Forms in Culture.

Now, in our attempt to explain the development of *Trichomonas* from a minute plasmodial form to a fully developed one, as observed by us, on basis of the above opinions, we suffer under the disadvantage that no observation based on phenomena similar to ours have been recorded by any other observer in any of the intestinal flagellates. For this reason we will have to fall back on the binucleata like *Herpetosoma*, *Crithidia* and *Trypanosoma* for comparison, as the phenomenon of multiplication in *Trichomonas*, as observed by us, resembles very much those found in these flagellates.

It will facilitate matters, if we make our position clear by defining the terminology which we are going to use in this paper in making our observation.

The anterior flagella of a *Trichomonas* will be designated by us as anterior directed axoneme or shortly as anterior axoneme, and the structure from which these originate will be designated as basal granules. The axostyle will be designated as posteriorly directed axoneme or posterior axoneme, as we will show later on that we found facts in support of views held by observers like Reuling and Alexieff, that axostyle is of flagellar origin and not a skeletal structure originating from paradesmose as held by Wenyon.

Origin of the Basal Granules and Anterior Axonemes (Flagella).

Jameson believes, as can be seen from the above quotation, that basal granules are offshoot from the karyosome and serve the function of centriole and are in fact a centriole. Kofoid and Sweezy in their description of the neuromotor apparatus of *Chilomastix*, describe an intra-nuclear blepharoplast (basal granule) and also a separate centriole situated in the karyosome of the nucleus connected with extra-nuclear blepharoplast (basal granule) from which the flagella take their origin. Doflein views the basalkörn as of independent origin, and behaves as a centriole. It may take its origin from the plasma itself. Now, if we define a centriole (centrosome) as energy centre from which the axoneme (axile fiber) of the flagellum takes its origin, we can suppose in explaining

the various types of organisms found in our culture, that this centriole can be situated either inside the nucleus, or outside it. Its position in the cell is an accidental factor, as stated by Wilson, and a Protozoa can possess a centriole both in the nucleus as well as outside it. In fact Hartmann gives a scheme for attachment of flagella to centriole in different types of flagellates. In one type it is intra-nuclear, in another it is situated in the nuclear membrane, and in another again outside the border of the nucleus. So the naming by Kofoid and Sweezy one centriole as intra-nuclear blepharoplast, another as centriole, another as extra-nuclear blepharoplast, has no legs to stand upon.

Next point to be decided is whether centriole is of chromatic origin. Wenyon is of opinion that there is a tendency to believe that karyosome is composed of achromatic substance. Doflein states, " Als Caryosome wären demnach solche Binnenenkörper von Kernen zie difiuieren, welche chromatinfrei sind, nur die Bewegungs substanz enthalten, welche bei der kernteiling sich in die Spindel unwandelt und so zur teilung in 2 Tochterkerne und zur verteiling der Trager der Erbsubstanzen, der Chromosomen, auf diese die nötige Bewegungsenergie liefern." But at the same time all authorities including the above two agree in defining karyosome as a centrally situated body and characterised by taking intense chromatic stain.

Doflein tries to explain these and other contradictory opinions by observing that we must remember that the structures which we see and describe after fixation and staining are not what are actually present in substance of the living Protozoa—they being of colloidal nature liable to change under varying circumstances and hence the exact use of terminology to different structures found after fixation is not applicable or justifiable.

The only way to get out of this difficulty is, if we assume that the centriole, whatever may be its origin, is dependent for its source of energy on the "Erbsubstanz," namely, the chromatic element which is differentiated from other elements of the Protozoa by its power of taking up chromatic stain. Applying this to our findings, it is to be presumed that in the minute nonflagellate stage of *Trichomonas* (Figs. 37 and 54) the centrioles from which the axonemes (both anterior and posterior) originate are situated within the nucleus. The single chromatic line found starting from the

centriole passing anteriorly, curving round and then forming the margin of the organism, is to be taken as an undifferentiated axoneme and a similar fiber directed backwards is the undifferentiated posteriorly directed axoneme (axostyle). Fig. 31 represents the next stage; in this the centrioles have taken up a position anterior to the nucleus, but they have not become separated as yet—the centrioles from posterior axonemes are presumed to be still inside the nucleus. In Fig. 58, the centrioles for anterior axonemes are presumed to be still inside the nucleus. In Fig. 39 the centrioles for anterior axonemes have become further differentiated and some of the centrioles from posterior axoneme have taken up a position anterior to the nucleus. The centriole of the posterior axonemes is not single; it differs in different species of *Trichomonas*, Reuling found in *Trichomonas vaginals* four fibers, and so it is presumed that the centrioles from these are at least four in number in the species. It may happen that some of the centrioles from the posterior axonemes have taken up their position anterior to the nucleus while some are still situated within the nucleus. In Fig. 46 the centrioles from both anterior and posterior axonemes have become definitely differentiated into a separate structure—the basal granules—the connecting fiber between the centrioles and the karyosome (centrosome) can be seen in Fig. 49 and have disappeared in Fig. 47.

Origin and Devolopment of Axostyle.

Next point to be dealt with is the origin and development of the axostyle. Though the origin has been incidentally dealt with along with the account of the origin of the flagella, yet it is necessary to deal with it in a little more detail on account of the conflicting and widely divergent views held by different authorities about its origin and function, an idea of which we have already given in the introduction of our paper.

Alexieff, by comparing the structures of the spermatozoa with that of the flagellates, has made out certain points which are worth quoting in this connection. " L'axostyle est une formation tubuleuse au filamenteuse qui est disposée suivant l'axe longitudinal du corps die Flagellé et fait souvent une saillie a l'exterieur (pointé caudale) chez les Flagelle's diplozoaires (*Octomitus*,

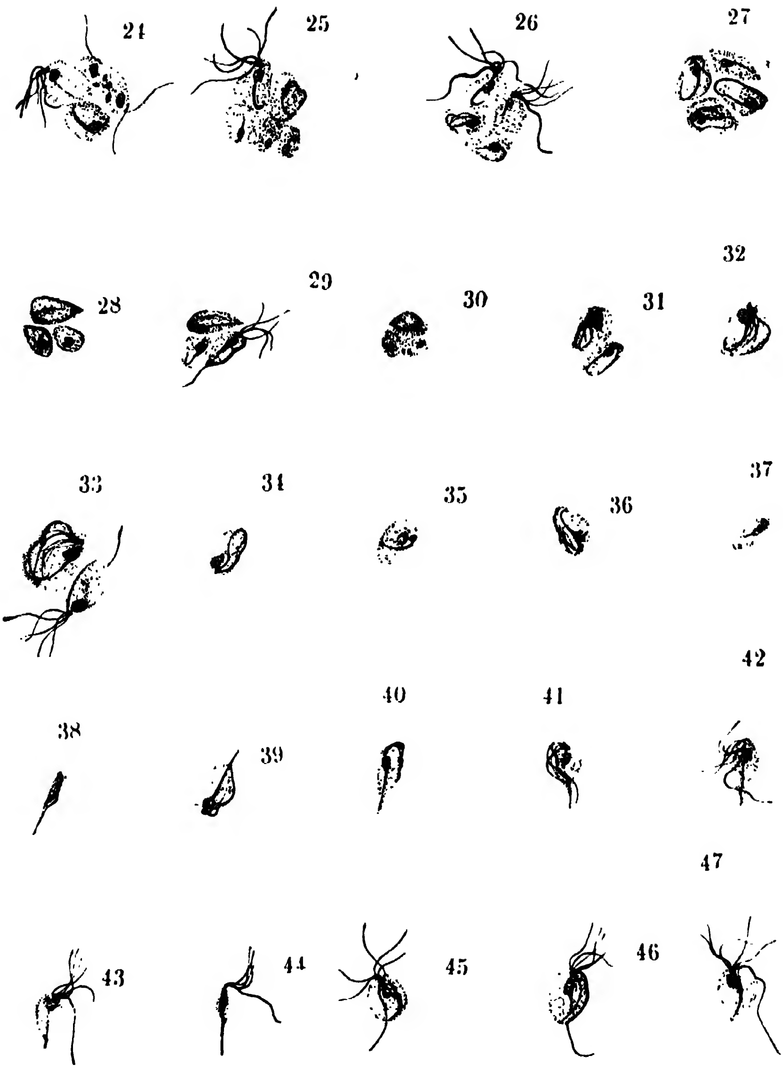


Plate II

Giradia et autres) cheque axostyle se prolonge avec un flagelle caudal............

2. Les proprietes tinctoriales de l'axostyle sont celles d'un flagelle.

3. L'axostyle joue le róle d'un gouvernail, c'est la fonction principale; en se contractant il se recourbe comme le font les flagelles et de cette façon it détermine le changement d'direction die Flagellé. Un role secondaire—meintenir la forma du corps constante, (roled' un quelette ínterne),...........

4. Pendant la division die Flagelle l'axostyle ancien disparait et deux nou veaux axostyles se forment aux dépens des blépharoplasts—fils et tout au debut tout a fait l'aspect de flagelles.

5. Etant donn'e qu'on note beaucoup de caracteres commune á axostyle et an flagelle (la colorabilete indentique, la contractilite le méme mode deformation), on doit admittre que l'axostyle est homologue die flagelle intra-cytoplasmique, c'est-a-dire un flagelle qui au lieu d étré libre est plongedans le plasma.

6. Entre l'axostyle des Flagellés et le filament axile des spermatozoi des it y a une homologie complete, le filament axile apparait comme un flagella á parter du blepharoplasts qui est homologue du centrosome."

Wenyon says, "The axoneme of *Hexamitus* often referred to as an axostyle but there seems no reason to suppose that they are homologous with the axostyles of *Trichomonas*. In *Hexamitus*, the axonemes usually stain deeply, while in *Trichomonas* the axostyle does not readily stain." Kofoid and Sweezy (1915a) are of opinion that an axostyle of *Trichomonas* represents the axoneme of a backwardly directed flagellum, as in *Hexamitus*.

Wenyon believes that during division of *Trichomonas*, after the nuclear membrane become constricted and divided, the divided blepharoplasts separate and the axostyle divides longitudinally from before the backwards. On the other hand Kuczynski (1914) from study of *Trichomonas muris* comes to the conclusion that old axostyle disappears, while new axostyles are formed as outgrowths from them. Wenrich is in agreement with Kuczynski.

We found that in the method of division of binary method which has been described in the latter part of the paper, the undulating membrane divides and the basal granules arrange into two groups and the flagella become duplicated and the nucleus divides in them,

but the axostyle does not disappear. It divides longitudinally. It does not disappear during any stage of the division (Figs. 5, 6, 7 & 8).

Now, in explaining the method of development of axostyle from our observation—in the earliest stage (Fig. 59), it is seen as a posteriorly directed axoneme originating from a centriole situated in the karyosome, its origin being similar to the anteriorly directed axoneme. In this condition as well as in later stages of its development it takes stain like an ordinary flagellum. In later stages of its development the centriole of the posteriorly directed axoneme, becomes separated along with the centriole of the anteriorly directed axoneme and occupies a position anterior to the nucleus adjacent to the centriole of the anteriorly directed axoneme. In this condition, the karyosome may be connected with the centriole by rhizoplast or these may disappear. The posterior axonemes may be represented by a single fibril or several fibrils. Besides, a condition may arise in which the centrioles of anterior axonemes may get separated themselves from the karyosome, but the centriole of posterior axoneme may not have separated itself and is still situated in the karyosome. Under this circumstance various appearances are seen which seem unexplainable at first, but which become clear when we remember the method of development.

After a time, in more developed *Trichomonas*, we find that the several axonemes comprising the axostyles originating from centriole situated anterior to nuclei, unite into a band, pass backwards and become tapering. Even in this condition it takes stain. Gradually the axonemic function of the posteriorly directed axoneme is replaced by skeletal function when it ceases to take stain.

Development of Undulating Membrane.

All observers have described the only way of formation of undulating membrane of a *Trichomonas*, is by division of the pre-existing membrane. The origin and development of undulating membrane from chromatic line which shows no trace of undulation, as we see in our specimen, have not been described in *Trichomonas* by any observer, though this has been found in *Crithidia* and *Trypanosoma*. In Fig. 52 will be seen the first indication of a rudimentary undulation. It is

very difficult to believe that the highly developed undulating membrane of adult *Trichomonas mabuia* with deep undulations can develop from a rudimentary stage such as seen in Fig. 57. Plate III will show all the stages in the growth of undulating membrane.

Method of Division by Binary Mitosis.

Binary divisional forms.—Longitudinal division by mitosis, besides the method of multiplication by gemmules, has been observed by us in specimens taken directly from gut or in cultures of 24 hours' standing. But these occurrences are very rare. The method of division is as follows:

The organism becomes rounded in shape. The undulating membrane surrounds nearly the whole of the body of the organism, except small portion at the posterior end (Fig. 5). Division of the basal granules synchronises with the division of the undulating membrane (Fig. 6)—in this stage, two groups of flagella originating from two groups of basal granules are seen, and also two undulating membranes. The axostyle does not disappear as yet, as is depicted by Wenrich in his description; but has not as yet divided into two. The nucleus has divided into two groups, from each of which three pairs of anterior axonemes originate. In Wenrich's description of division, the position of the basal granules are on the opposite side of the longitudinal axis of the dividing organism (anterior and posterior) as well as on the opposite sides of the transverse axis. In our case they occupy the same side, namely, the anterior end of the longitudinal axis, and on two sides of the transverse axis. In Hartmann's description of division of *Trichomonas muris* (*loc. cit.* p. 54), the two units formed from division, occupy position antero-posteriorly, on the same side of the transverse axis. The undulating membranes are on the same side of the axis of the division, whereas in our case they occupy two sides (right and left) of the axis of the division.

Explanation of some of the phenomena observed by previous observers but which have hitherto not been explained properly—Chatton's finding of Eutrichomonad forms (Trichomastix forms), in showing four free flagella without any attached one in cultures of ordinary *Trichomonas* has not been explained by any one. He has

inoculated these into peritoneal cavity of guinea pigs—these have developed into ordinary *Trichomonas*. This can be explained by our observations in which we found numerous Eutrichomonad forms in our culture of *Trichomonas*. In consonance with this view, in case of *Pentatrichomonas*, organisms showing six free flagella are seen (Fig. 43). Finding by Castalleni of an organism showing undulating membrane, in which no flagella were found in stool of man, and which he by mistake designated as *Entamœba undulans* is explainable by a cultural form of *Trichomonas*. Similarly, the small *Trichomonas* (about 6 μ) figured by Wenyon cannot be explained by any other way than by a cultural form.

Besides, as these cultural forms occur in intestine, many flagellates, described as new species, will be found to be cultural forms of *Trichomonas* and *Pentatrichomonas*, if scrutinised properly.

Conclusion.

The following conclusions can be drawn as the result of our observations which have been given in this paper.

(1) One method of multiplication of *Trichomonas* is by binary division.

(*a*) In this the division is longitudinal, from front backwards.

(*b*) Axostyle persists during the process of division and does not disappear at any stage of division.

(2) Another method of multiplication is by somatella formation with gemmation.

(*a*) In this we found minute gemmules containing the nucleus and undifferentiated axonome, these showing no other structures—all the gradation from this to the fully developed ones have also been found by us.

(*b*) From the study of the gemmules in different stages of development, one can make out very clearly the various stages of development of:

(*i*) flagella, from a single forwardly directed axoneme originating from a centriole situated in the karyosome ;

(*ii*) axostyle, from a posteriorly directed axoneme also arising from a centriole situated in the karyosome, giving rise to the axostyle.

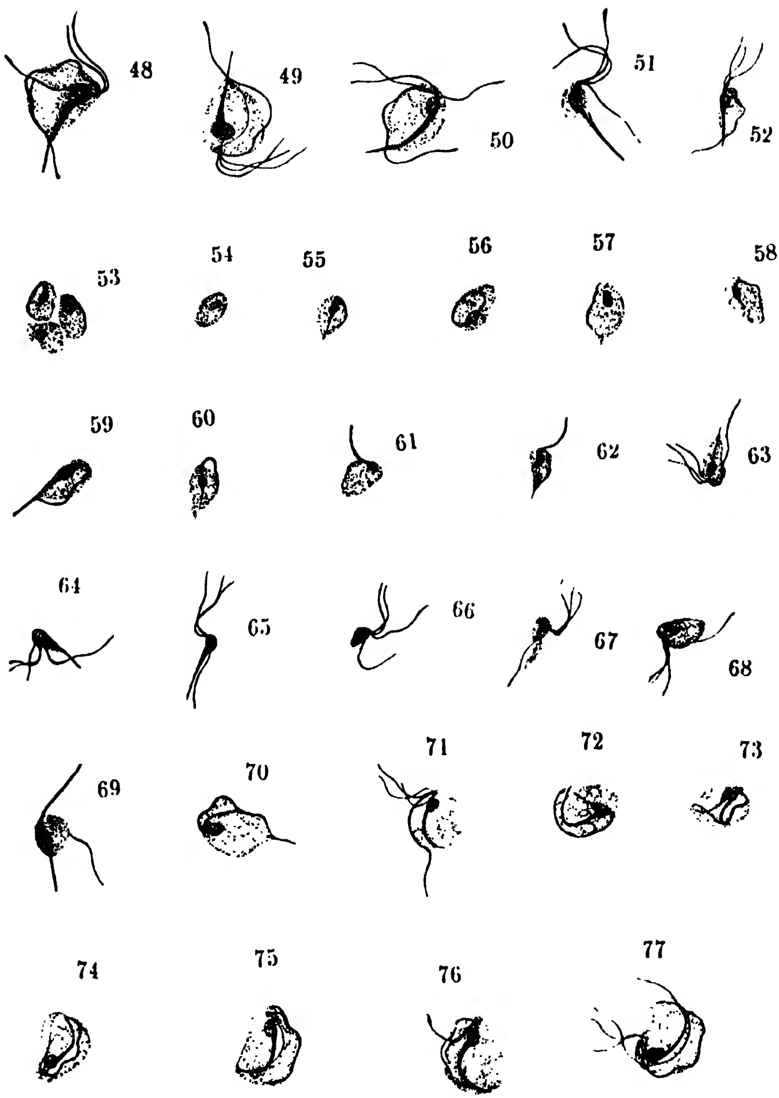

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PLATE III.

EXPLANATION OF PLATES.

Figures in Plates I and II are from cultures of *Pentatrichomonas bengalensis* drawn under Camera Lucida, oil immersion $\frac{1}{12}$ objective and No. 7 eye-piece, stained by modified Leishman. Figures in Plate III are from cultures of *Trichomonas mabuia* drawn under the same condition, mentioned above, stained by modified Leishman.

PLATE I.

Figs. 1 to 3.	Adult *Pentatrichomonas.*
Figs. 4 to 8.	Stages in binary division of *Pentatrichomonas.*
Figs. 9 to 11.	Young forms.
Figs. 12 to 15.	Binary divisional forms of young forms found in culture.
Figs. 16 & 17	Somatella forms with several nuclei and basal granules, provided with temporary flagella.
Figs. 18 & 19	Somatella forms with a gemmule showing 5 free flagella and an attached one and the faintly stained temporary flagella of the somatella.
Fig. 20.	Somatella, in which the gemmule shows flagella which are intra-cytoplasmic.
Figs. 21 to 23.	Nearly separated gemmules.

PLATE II.

Figs. 24 to 26 and 29 & 33.	Flagellated gemmule and plasmodial forms just separated from the somatella.
Figs. 27, 28, 30 & 31.	Plasmodial forms.
Figs. 32, 34 to 37.	Plasmodial forms during development.
Figs. 38 to 40.	Different stages of development of undulating membrane of young forms.
Fig. 41.	Young form in which the flagella have become free.
Figs. 42 & 43.	Young form in which all the flagalla are free, Eupentatrichomonas form.
Fig. 44.	Bacillary form.
Figs. 45 to 47.	Young flagellated forms.

Plate III.

Figs. 48 to 50	Different forms of *Trichomonas* from Mabuia carinata.
Fig. 51.	Eutrichomonas form.
Fig. 52.	Young form of the *Trichomonas lacertae.*
Fig. 53.	Plasmodium form just separated out from the somatella.
Figs. 54 to 58.	Plasmodial forms with outgrowths of axonemes from the nucleus.
Figs. 59 to 62.	Same with their axonemes more developed.
Fig. 63.	Young *Trichomonas* just developed from a plasmodium form.
Figs. 64 to 67	Minute bacillary forms.
Figs. 68 & 69.	Young forms of *Trichomonas lacertae.*
Figs. 70 to 77	Cultural forms of the same.

REFERENCE.

1. Chatton, E. (1918 b)—Culture pure indefinie d'un flagelle intestinal du genre Trichomastix sur organes en autolyse aseptique. C. R. Soc. Biol., Vol. LXXXI, p. 346.

2. Chatterjee, G. C. ('927)—A note on the Method of Multiplication of Trichomonas flagellates of different species in artificial culture. Cal. Univ. Jour. Sci., Vol. VIII, p. 15.

3. Dobell, C. (1910)—On Some Parasitic Protozoa from Ceylon. Spolia zeylanica, Vol. VII, p. 65.

4. Dobell, C. and O'Connor. (1921)—The Intestinal Protozoa of Man. London, p. 69.

5. Doflein, F. (1922)—Untersuchungen uber Chrysomonadinen. Arch. Protist., XLIV, p. 149.

6. Hartmann, M. & Schilling, C. (1917)—Die pathogenen Protozen und die durch sie verursachten Krankheiten. Berlin.

7. Jameson, A. P. (1914)—A New Phytoflagellate (*Parapolytome satura* n.g., n.sp.) and its method of nuclear division. Arch. Protist., XXXIII, p. 21.

8. Kofoid, C. A. & Sweezy, O. (1915 a)—Mitosis and Multiple Fission in Trichomonas flagellates. Proc. Amer. Acad. Arts and Sci., LI, p. 289.

9. Kuczynski, M. H. (1914)—Untersuchungen an Trichomonaden. Arch. Protist., XXXIII, p. 119.

10. Lynch, K. M. (1913 a)—Trichomoniasis of the Vagina and the Mouth : cultivation of the causative organism and experimental infection. A preliminary communication. Amer. Jour. Trop. Dis. and Prev. Med., II, p. 627.

11. Reulings, F. (1921)—Zur Morphologie von Trichomonas vaginalis Donne. Arch. Protist., XLII, p. 347.

12. Tanabe, M. (1926)—Morphological studies in Trichomonas. Jour. Parasit. Urbana. Vol. XII, p. 120.

13. Wenyon, C. M. (1926)—Protozoology. London.

14. Wilson, E. B. (1911)—The Cell in Development and Inheritance. New York.

The Purana Rocks of the Indian Peninsula—a Study in the Petrologic Method of Correlation

BY

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Introduction.

The recent identification of the Cambrian brachiopod *Acrothele* [1] among the Vindhyan materials collected many years ago by Mr. Jones [2] has produced a new interest in the Purana rocks of the Indian Peninsula and the present paper embodies the results of an attempt at correlating the different exposures of the Purana rocks as found in peninsular India. It may be pointed out that there are some who think that the fossil belongs to the plant kingdom.[3]

Fixation of a Datum-line.

As far as we know at present India has passed through important periods of volcanic activity, namely, the Dharwar, the Cuddapah, the Rajmahal and the Deccan Trap with fairly extensive lava flows. The peninsular Purana rocks are scattered over a wide area and, in many cases, they have been found associated with volcanic rocks and, as there are no inherent grounds for doubting that the volcanic rocks associated with these Purana formations are contemporaneous, in all that follow here these volcanic rocks have been taken as the standard of reference.

Two Phases of the Purana Flow.

It has been shown by the author of this note[4] that in the city of Bombay there are evidences showing that the Deccan trap lava follows the rule of increasing acidity while the statement of Dr. Fermor [5] that

[1] General Report for 1926, *Rec. Geol. Surv. Ind.*, LX, 18, 1927.
[2] *Proc. Asiat. Soc. Bengal*, CVII, 1908.
[3] *Proc. Ind. Sci. Congress* (16th), 257, 1929.
[4] *Journ. Dep. Sci., Cal. Univ.*, VIII, 123-128, 1927.
[5] *Rec. Geol. Surv. Ind.*, XXXIV, 148-166, 1906.

the Pavagad flow is alternately acidic and basic has been questioned by Prof. Mathur [6] according to whom the Pavagad section shows evidence of increasing acidity as one proceeds upwards. My own observations in the Bhavnagar State (Kathiawar) also confirm the opinion of Prof. Mathur. For a very detailed account of the Purana rocks as found in the Son valley we are indebted to Messrs. Oldham, Vredenburg and Datta.[7] A study of this account shows that rocks of the Bijawar age are overlain by the lower Vindhyans with an intermediate series of red shales (the Jungel series) lying between the two. The Bijawars are interbedded with basic lavas and tuffs; the immediately overlying Jungel series has no volcanic element in it while the lower Vindhyan contains, in its lower half, the Porcellanite stage. This stage, according to Mallet, was supposed to consist of an intermediate member of 'Trappoid beds' both underlain and overlain by porcellanic shales,[8] but, after a careful study of the rocks of this stage, Vredenburg found that the rocks were all practically identical and recorded the following observations:—

"In conclusion it may be said that the 'trappoids' and 'porcellanites' that form a large portion of the lower Vindhyan strata and whose true nature has remained for a long time an undecided question, are rhyolitic tuffs of varying coarseness......It may be concluded from this distribution that an extensive series of acid eruptions began simultaneously with, or just before the commencement of the Vindhyan era. At the time of maximum activity, showers of fine volcanic dust together with coarser debris were ejected in enormous quantity giving rise to the tuffs forming the 'porcellanic stage.' The beds occasionally found at higher horizons may represent stray eruptions belonging to a period of declining activity" (*op. cit.* pp. 107-108).

Attention may be drawn in this connection to the rock group known as the Malani Volcanic series so elaborately described by La Touche.[9] The Malani Volcanic series has been found to lie between the rocks of the Aravalli system and the sandstone of the upper Vindhyan age. The rocks comprising the Malani Volcanic series are

[6] *Proc. Ind. Sci. Congress* (15th), 291, 1928.
[7] *Mem. Geol. Surv. Ind.*, XXXI, 1-178, 1901.
[8] *Mem. Geol. Surv. Ind.*, VII, 1-129, 1869.
[9] *Mem. Geol. Surv. Ind.*, XXXV, 1-116, 1902.

chiefly rhyolites which 'in many places,...are interstratified with thick beds of tuff and breccia formed out of the dust and fragments of the lavas themselves, and evidently due to volcanic explosions.' Rocks of a more basic type are also found associated with these acid lavas, but it is not definitely known if these basic rocks are interbedded or intrusive though 'on the whole...the balance of evidence is in favour of their being intrusive.' The general lie of the Malani series and its predominating acidic characters have led me to conclude that the rocks of the Malani series represent the acid lavas which have been described from the Son valley and are hence lower Vindhyan in age. Reference may also be made in this connection to the presence of a small thickness of red shales found immediately below the Malani rocks and to the red shales of the Jungel series found immediately below the lower Vindhyan of the Son valley to which reference has already been drawn. It appears from all that have been stated above that the Purana lava flows in the peninsular portion of India may be considered as having passed through two different phases, *viz.*, basis and acidic, and that the basic flow is older than the acidic. The basic flow was typically developed in the lower part of the Cuddapah series and the acidic flow in the lower Vindhyans.

Cuddapah Series.

Among the rock-formations of the Purana group found in southern India the most important are those known as the Cuddapahs and the Karnuls described by King. They are typically developed in the districts bearing their names and the junction between them is one of a marked unconformity.[10] Four members of the Cuddapahs have been recognised and they are the Papaghni, the Cheyair, the Nallamalai and the Kistna in an ascending order. Sir Thomas Holland proposed a twofold division of the Purana rocks as developed in peninsular India.[11] The lower division included, among other beds, the lower half of the Cuddapahs, *i.e.*, the Papaghni and the Cheyair while the upper division included 'the original Vindhyans, the Bhima series, the upper part of the Cuddapahs and the Karnools.' The presence of the

[10] *Mem. Geol. Surv. Ind.*, VIII, 1-313, 1872.

[11] Geology of India, *Imp. Gazetteer*, I, Ch. II, 61, 1904.

basic lava flows in the lower division and their absence from the upper were considered to be the chief characters of these two divisions by Sir Thomas Holland. This classification was a little modified by the late Mr. Vredenburg[12] who divided the Cuddapahs into two divisions, the lower and the upper and Wadia, in his text-book,[13] followed him. These two authors have, however, specified no reasons for this twofold classification of the Cuddapahs. It appears to me that this twofold division of the Cuddapahs is quite justified when we bear in mind the important lithological distinction that exists between the two divisions inasmuch as indications of igneous activity which are to be noticed in the lower division are quite absent from the upper, a feature to which, as already noted, attention was drawn by Sir Thomas Holland. Hence we may consider the Cuddapahs to be divided in the following manner :—

Cuddapah	Upper	Kistna	Srishalam quartzite.
			Kolamnala slate.
			Irlakonda quartzite.
		Nallamalai	Cumbum slate.
			Bairenkonda quartzite
	Lower	Cheyair	Pullampet slate.
			Nagari quartzite.
		Papaghni	Vempalli slate.
			Gulcheru quartzite.

As it has been already mentioned the evidences of igneous activity during the Cuddapah time are all confined to the lower portion and the geological map of the Cuddapah and the Karnul formations shows this very clearly. From a study of King's description of the formations [14] it appears that the Gulcheru quartzites are not associated with any trap flow; the Vempalli

12 A Summary of the Geology of India (2nd edition), 29-31, 1910.

13 Geology of India (2nd edition), 73, 1926.

14 *Mem. Geol. Surv. Ind.*, VIII, 1-313, 1872.

slates 'are largely associated, nearly in the stratification, with intrusive trap,'[15] the Nagari quartzites 'are traversed by intrusive trap, which, in the flows, has deviated only slightly from the lie of the strata'[16] while a series of contemporaneous flows has been met with in the Pullampet slate. Besides these flows, the Pullampet slates contain two other rocks to which a volcanic origin has been attributed, namely a group of mottled and flaky shales and bands of felsite. The mottled and flaky shales are supposed to represent beds of ashes, while the chert-like felsites have been found to be associated with the trap flows. It may be mentioned that these chert-like silico-felspathic bands have not been properly studied as yet and hence it is difficult to pronounce any opinion about them. The nature of the traps has been very carefully studied by King according to whom there were two periods of volcanic activity during the Cheyair time : 'the first and strongest early in the formation of the series ; the next, much less extensive and about two-thirds up in the series.'[17] The relationship of the traps to the surrounding country-rock was also carefully gone through and the conclusion drawn was that 'considered generally, the traps are mostly contemporaneous, partly intrusive.'[18] There is no trace of any igneous activity in the upper part of the Cuddapah formation and in the Karnuls.

The Upper Cuddapah and the Lower Vindhyan.

I have already given my reasons for holding that the Malani rhyolites and the members of the 'Porcellanite' stage of the lower Vindhyan represent the later phase of the basic Cuddapah flow while the occurrence of a silico-felspathic or felsitic rock in the Pullampet slates, as noted above, may be taken to establish that even in the Cuddapah area, the lower basic flow was differentiated, though possibly very locally, into an upper acidic one. It is also clear that during the flow of the acidic lavas, the upper part of the Cuddapahs was free from any volcanic activity, the seat of the activity having migrated towards the north. The lower and the upper parts of the Cuddapah

[15] *Op. cit.*, p. 160.
[16] *Op. cit.*, p. 169.
[17] *Op. cit.*, p. 197.
[18] *Op. cit.*, p. 203.

are but two divisions of one big geological system, *i.e.*, the Cuddapah. The Cuddapah and the Malani flows which are but two phases of the same magmatic activity should not be considered as being separated from each other by a time-interval which corresponds in value to a geological system. Hence the only conclusion that may be drawn is that during the upper Cuddapah time there was no volcanic activity in the Cuddapah area, but that a later phase of the basic flow of the lower Cuddapah beds is represented by the acidic flow of the lower Vindhyan and hence the lower Vindhyan should be correlated with the upper Cuddapah and not with the Karnul as is generally supposed.

Karnul and Vindhyan.

While describing the Vindhyan system Mallet [19] expressed the opinion that the Vindhyan system of Central India represented the Karnul beds of southern India. Mallet, however, did not definitely state whether by the expression Vindhyan he wanted to indicate the lower Vindhyan, the upper Vindhyan or the whole of the Vindhyans, but it appears more likely that Mallet's idea was to correlate the upper Vindhyans with the Karnuls. It may be mentioned also that Medlicott [20] in 1882 concluded that the Cuddapahs represented the lower Vindhyans, but this opinion was not supported by Oldham in his Manual of Indian Geology.[21] The opinion accepted at present appears to be to look upon the Karnuls as being equivalent to the lower Vindhyans.[22] I have already given reasons for holding that the lower Vindhyans correspond in age to the upper Cuddapahs. This conclusion which is based upon the nature of the volcanic flow gets an additional support from the fact that the lower Vindhyans show evidence of crustal movement like the members of the Cuddapahs. The beds of the upper Vindhyan are, as a rule, nearly horizontal like those of the Karnul series showing that they were not affected by the orogenic movements of the Cuddapah time. I think that, in the absence or scarcity of fossils, this structural peculiarity should be considered as of great importance in the matter of correlation. The occurrence of

[19] *Mem. Geol. Surv. Ind.*, VII, 126, 1869.
[20] *Rec. Geol. Surv. Ind.*, XV, 2, 1882.
[21] *Op. cit.*, p. 108, 1893.
[22] *Mem. Geol. Surv. Ind.*, LI, Pt. I, 107, 1926.

diamontiferous beds in the Karnul and in the upper Vindhyan also very suggestive in this connection.

Kaladgi Series.

The next group to be described, in this connection, is the Kaladgi series [23] which is divided into two parts, the upper and the lower. The upper part of the Kaladgi series consists of trap dykes and, from a general lithological consideration, it was suggested by Foote and King that the Kaladgi corresponded with the Cheyair group. Foote has also drawn attention to the lithological similarity between the Par beds and the lower Kaladgi [24] and, as I am going to suggest in the sequel, while dealing with the rocks of the Gwalior series, that the Morar group (upper Gwalior) and the Par group (lower Gwalior) possibly represent the Cheyair and the Papaghni stages respectively, it may be concluded that the upper Kaladgi series corresponds with the Cheyair and the lower Kaladgi series with the Papaghni.

Bhima Series.

The rocks of the Bhima series are developed in the neighbourhood of the Kaladgi series without ever coming into contact with them. Foote [25] suggested the comparison of the Bhima series with the Karnul series chiefly on lithological grounds, the only possible resemblance between the two being the absence of any contemporaneous flow from both of them. The rocks of the Bhima series have generally got their horizontality disturbed in a rather marked way while Prof. Sampat Iyengar [26] has described a section in which the Bhima rocks were observed to lie in a gentle anticline. As far as we know the Karnuls are usually horizontal or deviate only slightly from the horizontal and hence it is doubtful if the Bhimas can be correlated with the Karnuls and I think that a more reasonable solution will be to correlate the Bhima rocks with the upper Cuddapahs.

[23] *Mem. Geol. Surv. Ind.*, XII, 70-138, 1876.
[24] *Op. cit.*, p. 137.
[25] *Mem. Geol. Surv. Ind.*, XII, 164, 1876.
[26] *Proc. Ind. Sci. Cong.* (11th), 164, 1924.

The Pakhal and the Sullavai Series.

The Purana rocks of the Godavari-Pranhita valley are known as the Pakhal and the Sullavai series. Foote put the two subdivisions of the Pakhal series—the Pakhal and the Albaka—in a line with the Nallamalai and the Kistna respectively and there are no grounds for demurring to this opinion,[27] *i.e.*, they correspond to the upper Cuddapah or the lower Vindhyan according to the scheme of correlation outlined above. The Sullavais which have been found to overlie the Pakhals [28] have been correlated with the Karnuls. This correlation has been further borne out by the similarity in the weathering of the Kapra sandstone (the uppermost member of the Sullavai series) and of the pinnacled quartzites of the Paneum group of the Karnul formation. There is no inherent objection to the acceptance of this view, but as I am of opinion that the lower Vindhyans are represented by the upper Cuddapahs and the Karnuls represent the upper Vindhyans, the Sullavais may be taken to correspond with the upper Vindhyans in the Godavari-Pranhita valley.

Bijawar Series.

The rocks of the Bijawar series are too well-known to deserve any detailed description here and the only feature of the rocks of this series to which attention should be drawn in connection with the present problem is the fact that 'trap rocks take a prominent part among the Bijawars.' [29] It has been pointed out by Sir Thomas Holland and Mr. Tipper that 'lithologically they (the rocks of the Bijawar series) resemble the Gwalior series and the Cheyair stage of the Cuddapah system.' [30] This opinion has been accepted in the present paper.

Penganga Series.

The Penganga or Pemganga (Premganga) beds are developed in the Pranhita valley. Regarding the correlation of these beds the following observations were recorded by Mr. Oldham :—[31]

[27] *Mem. Geol. Surv. Ind.*, XVIII, 212, 1881.

[28] *Ibid*, 227-236.

[29] *Mem. Geol. Surv. Ind.*, II, 43, 1859.

[30] *Mem. Geol. Surv. Ind.*, LI, 48, 1926.

[31] A Manual of the Geology of India, p. 90, 1893

'These Penganga beds were regarded by the earlier observers, and have always been referred to in the Survey publications, as of Vindhyan age......it may be noticed that the evidence in favour of identifying the Pakhal and Penganga beds with the Cuddapahs is as strong as it can be in the case of unfossiliferous rocks, where there is an absence of absolute continuity of outcrop. They were unhesitatingly identified by Dr. King who examined both areas. The general lithological resemblance is described as very close.' The Penganga beds, however, contain no trace of any volcanic activity and may hence be correlated with the upper Cuddapahs or the lower Vindhyans like the Pakhal beds as pointed out above.

Gwalior Series.

The next rock group to be considered is what is known as the Gwalior series. Hacket [32] divided the series into two groups, namely the Par and the Morar. The Par group is chiefly one of quartzites and free from any signs of volcanic activity, while the Morar group includes a number of trap flows of which there are four great spreads. The rocks of the Morar group include also ash-beds as well as thin and regularly bedded felsites. The true nature of these felsitic beds is not known, but it appears from the results of the chemical analysis published that these felsites may be of a trachy-andesitic nature, the percentage of silica being 60·50. Lithological considerations show that the Morar group represents the Cheyair stage of the Cuddapah and the Par group falls in a line with the Papaghni stage in both of which quartzites play a very important part. Attention may be drawn in this connection to the presence of felsites in the Pullampet slates (Cheyair stage) to which reference has already been made.[33]

Iron-ore Series.

This is a division of rock series recently recognised in Chota-Nagpur.[34] Dr. Dunn is of opinion [35] that the rocks of the Iron Ore

[32] *Rec. Geol. Surv. Ind.*, III, 33-42, 1870.

[33] According to Mr. Coulson, the rocks of the Gwalior, Bijawar, Cheyair and Papaghni series are of Dharwar age. He has, however, given no convincing arguments for entertaining this opinion. (*Rep. Austral. Assoc. Adv. Sci.*, XVIII, 316-317, 1926.)

[34] *Mem. Geol. Surv. Ind.*, LI, Pt. 1, 89, 1926.

[35] *Rep. Austr. Assoc. Adv. Sci.*, XVIII, 307-308, 1926. See also in this connection *Mem. Geol. Surv. Ind.*, LIV, 154-155, 1929.

series are Dharwarian in age. This question was discussed by Dr. Heron who has expressed himself as quoted below :—

"It appears to me that some of Dr. Dunn's difficulties of correlation might be got over by considering the Iron Ore series of Singhbhum as a post-Archaean formation ; there is admittedly a very pronounced uncomformity between it and the 'Older Metamorphics.' Dr. Dunn, however, states that this view is 'quite untenable,' arguing from the existence of diastrophism and metamorphism in North Singhbhum, and apparently putting aside the conflicting evidence in South Singhbhum where the Iron Ore series is unaltered and the unconformity between them and the underlying older metamorphics is 'very pronounced.' In the type area of the Cuddapahs, where the beds are little disturbed, intrusive traps occur and I cannot agree with Dr. Dunn in his contention that the Iron Ore series cannot be analogous (using the word in a wider and looser sense than 'contemporaneous') to the Cuddapahs, because in N. Singhbhum the Iron Ore series is much metamorphosed and intruded by igneous rocks."[36] I am in full agreement with Dr. Heron's views on the subject and wish to correlate the rocks of this series with the rocks of lower Cuddapah age, the correlation being based on the presence of the Dalma trap in the Iron Ore series. In his general report for 1927, Sir Edwin Pascoe has recorded the following observations regarding the age of the Iron Ore series.[37]

"The youngest of the formations (of the Iron Ore series) are basic igneous rocks, seen mostly as dykes in the granite region and as sills in the Iron Ore series......

"The petrographic types include gabbros, dolerites and basalts... There is a great deal of similarity between these and the trap dykes of Cuddapah age in the Madras Presidency, and the dykes in the Bijawars and Gwaliors...which suggests that all these may have a community of origin."

It may be pointed out, in this connection, that the beds which constitute the Iron Ore series were also described as being of the Cuddapah age by Mr. Maclaren.[38]

[36] *Ibid*, p. 310.

[37] *Rec. Geol. Surv. Ind.*, LXI, 98, 1928.

[38] *Rec. Geol. Surv. Ind.*, XXXI, 73, 1904.

Delhi Series.

There is a good deal of controversy regarding the age of the rocks of the Delhi series and the whole question has been summarised by Dr. Heron.[39] This series consists of the following :—

Ajabgarh series.
Hornstone breccia.
Kushalgarh limestone.
Alwar series.
Raialo limestone and quartzite.

According to Dr. Heron it is fairly certain that the Delhis are post-Dharwars and more probably "roughly contemporaneous with the Cuddapahs or the Bijawars" than older than these series. I am inclined to accept this view of Dr. Heron in a general way and think that the Delhi series practically embraces the whole of the Cuddapah series. This opinion is based on (*i*) the presence of the Nitahar bedded trap at the base of the Alwar series and (*ii*) the interstratification of the Malani rhyolites with the sedimentaries of the Ajabgarh type in the Kirana hills. It may be noted that, according to Dr. Dunn, the Delhi series is Dharwarian in age,[40] but it appears that he himself recognises that he does not stand on very secure and non-challengeable grounds.

Conclusion.

The correlation of the main members of the Purana group of peninsular India has been dealt with in the preceding pages and the result is summarised in the table annexed hereto. It may be mentioned that the basis of the whole scheme is the suggested relationship between the Cuddapah flow and the Malani rhyolite. It has also been pointed out that the conclusions arrived at are not entirely new as suggestions which may be compared with the main conclusions that the upper Vindhyans are correlated with the Karnuls and the lower Vindhyans with the Cuddapahs were already made by some eminent members of the Indian Geological Survey, namely Mallet

[39] *Mem. Geol. Surv. Ind.*, XLV, 105-116, 1925.

[40] *Rep. Austr. Assoc. Adv. Sci.*, XVIII, 302-304, 1928. It may be noted in this connection that Dr. Fermor is also inclined to treat the rocks of the Delhi system as being of Dharwarian age (*Rec. Geol. Surv. Ind.*, LXII, 395, 1930).

and Medlicott. Reference has also been made to three papers published very recently, two by Dr. Dunn and one by Mr. Coulson. One of Dr. Dunn's papers deals with the classification of the Archaean rocks of peninsular India, while Mr. Coulson's paper is of a more comprehensive nature as it deals with the rocks of the Archaean (gneisses and crystalline schists as also the Dharwar system) and the Purana group. Dr. Dunn has put the Iron Ore series and the Delhi series under the Dharwars while Mr. Coulson has included the Gwalior, Bijawar, Cheyair and Papaghni series as Dharwars and completely omitted the Karnuls. These are the views with which, as already noted, the author of the present paper is not in agreement.

TABLE.

Correlation of the Purana Rocks of Peninsular India.

Southern India.				Central Provinces.	Gwalior.	Chota-Nagpur.	N. E. Rajputana.	Western Rajputana.	Central India.
Karnul			Sullavai					Jodhpur Sand-stone	Upper Vindhyan
Cuddapah (Upper): Kistna, Nallamalai		Bhima	Pakhal system: Albaka, Pakhal	Penganga			Delbi series	Malani Rhyolite	Lower Vindhyan
Cuddapah (Lower): Cheyair, Papaghni	Kaladgi: Upper Kaladgi, Lower Kaladgi				Gwalior: Morar, Par	Iron-ore series			Bijawar

On the occurrence of Upper Palaeozoic fossils in the vicinity of Solon (near Simla).

By

Hemchandra Das-Gupta, M.A., F.G.S.

(With Plate I)

Introduction.

In a recently published Memoir [1] of the Indian Geological Survey, Dr. Pilgrim and Mr. West have recorded the results of their study of the structure and correlation of the Simla rocks and put the series of beds from the Blaini Conglomerate to the top of the Krol sandstone under the lower Gondwana which is practically equivalent to Carbo-lower Triassic.[2] About 12 years ago, a paper under the joint-authorship of Vredenburg and Das-Gupta [3] was read at the fifth session of the Indian Science Congress held at Lahore announcing the discovery of *Chonetes* sp. in Krol beds but, owing to certain difficulties for a short account of which a reference may be made to a previous communication [4] of mine, the paper read at Lahore could not be published. This paper was divided into two parts, the first part was communicated by the late Mr. Vredenburg and the second by the present author. The first part dealt with the general consideration of the question about the correlation of the pre-Tertiary formations of the southern Himalayas on account of their generally unfossiliferous condition and the second part dealt with the description of the fossils chiefly *Chonetes* and the conclusion that could be drawn from them. I do not know anything about the fate of the first part of the paper and, since the conclusion arrived at by Dr. Pilgrim and Mr. West has some bearing on that

1 Mem. Geol. Surv. Ind., Vol. LIII, pp. 1, 140, 1928.
2 Proc. Ind. Sci. Cong. (14th), p. 240, 1927.
3 Journ. Asiatic Soc. Beng., N. S., Vol. XIV, Proc. clxxxv (Abst.), 1918.
4 Journ. Dept. Sci., Cal. Univ., VIII, pp. 1-14, 1926.

of mine, I think it proper to publish the main part of my paper together with a photograph of *Chonetes* sp. Pilgrim and West are in doubt regarding the age of their Krol Series (*i. e.*, the part of the Krol beds overlying the Krol sandstone). In the classificatory scheme published by them, the age of the series has been left undecided, but it has been suggested that it is not impossible that the age may be Mesozoic. The results of my study brought me to the conclusion that the fossil in question was a species of *Chonetes*, it had been obtained from the Krol limestone (lower) and that hence the lower portion of the Krol beds was upper Palaeozoic, *i.e.*, the lower Gondwana bracket of Dr. Pilgrim and Mr. West as given at p. 3 of their Memoir should be extended further up to include, at least, the lower part of the Krol Series. The point can only be decided by a critical examination of the fossil and hence the necessity of the publication of this paper after the lapse of more than a decade.

Description of the Fossils.

In the year 1915, I had an opportunity of visiting a part of the country in the neighbourhood of Solon in charge of a party of Post-Graduate students from the Presidency College, Calcutta, the particular area being selected at the suggestion of (the late) Prof. Vredenburg and, in course of this visit, I happened to obtain the fossils described in this note at the junction of a small *nullah* with the Blini river, below Basal, lying S. W. of Basal and N. N. W. of Solon. The fossils were obtained in a bed of limestone corresponding to the lower limestone bed of the Krol Series.

The fossils that have been found are in a very bad state of preservation and referable to the brachiopoda and the gasteropoda. Among the brachiopod shells there is one the generic determination of which is doubtful while there is another which has been identified as *Chonetes*. This is represented by one valve (the ventral) and is without any ornamentations. No trace of the spines can be detected along the hinge-line and this might be due to the bad state of preservation or the species might not have had any spine at all.[5] Reference may also be made in this connection to *Chonetes aspinosa*

[5] Pal. Ind., Ser. XIII, Vol. I, p. 614, 1884.

PLATE I

Chonetes sp. (magnified).

described from the Carboniferous beds of New South Wales [6] which has no spines either on the surface or on the boundary of the cardinal area. The specimen that has been described as a gasteropod is one of the Capulidae and an *Orthonychia*.

As a result of the researches on *Chonetes* by de Koninck,[7] Waagen,[8] Tschernyschew,[9] and Girty [10] the following sections of *Chonetes* are recognised :—I. Concentricæ. II. Striatæ. III. Plicosæ. IV. Rugosæ. V. Læves. VI. Grandicostæ. VII. Striatocostæ and VIII. Pustolosæ. The species of *Chonetes* obtained from the neighbourhood of Solon can be unhesitatingly referred to the section of Læves and has got some resemblance with *Chonetes avicula* Waagen,[11] but the Himalayan species is more elongated transversely, while comparison with the available materials would seem to suggest that the species might be one hitherto undescribed. The breadth of the valve is 12 mm. and its length is 7 mm.

The fossil identified as *Orthonychia* is not folded longitudinally and is related to the group *Orthonychia Bohemica.*[12] On careful examination, the specimen seems to be very closely allied to *Orthonychia nuda* Spitz [13] described from the Carnic lower Devonian.

Age of the Fossils.

Chonetes ranges from Upper Ordovician to Permian, the Upper Ordovician forms being referable to the sub-genera *Eoderonaria* of Breger and *Eochonetes* of Dr. Reed.[14] One of the Permian species is *Chonetes granulifer* Owen which has been very elaborately studied by Greene. [15] In the lower Palaeozoic beds the genus is represented

[6] Rec. Geol. Surv., N. S. W., Vol. VII, pp. 69-71, 1902.
[7] de Koninck, Productus et Chonetes, p. 185, 1847.
[8] Pal. Ind., Ser. XIII, Vol. I, p. 615, 1884.
[9] Mem. du Comite. geologique, Tom. XVI, p. 229, 1902.
[10] U. S. Geol. Surv., Prof. paper 58, pp. 221, *et seq.*, 1908.
[11] Pal. Ind., Ser. XIII, Vol. I, p. 622, 1884.
[12] Syst. Sil., Vol. IV, Tom. 3, p. 134, 1911.
[13] Beitr. zur Pal. Osterr-Ung., Bd. XX, p. 162, 1907.
[14] Trans. R. S. Edin. Vol. LI, p. 916, 1917.
[15] Journ. Geol., Vol. XVI, pp. 654-663, 1908.

by a few species only, *e.g.*, those described by de Koninck [16] and Chapman [17] and they all belong to the *Striati* section. The Devonian representatives of the genus that I have been able to make out also all belong to the *Striati* section and, in course of his description of the Guadalupian fauna, Mr. Girty has observed that 'while the *striati* were persistent from the Devonian until the genus ceased to exist the *læves* I have come to regard as a subsequent development and as conditionally indicating rather late Carboniferous time. At least such seems to be the case in the American sequence known.[18] I think that the distribution of the Indian species of *Chonetes* is not inconsistent with the remarks just quoted. A few species of *Chonetes* belonging to the section of Læves have been described from the Productus-limestone beds and of the 5 species known one belongs to the lower, one to the middle, and three to the upper division of the Productus-limestone beds.[19] *Orthonychia* appeared a little later, *i.e.*, in Silurian but became extinct earlier, *i.e.*, in the Carboniferous. From these considerations one is tempted naturally to put the beds containing these fossils as belonging to the Upper Palaeozoic. Thus the Krol beds can be unhesitatingly assigned, in their lower portion at least, to the Upper Palaeozoic.

[16] *Op. cit.*, p. 227.
[17] Proc. Roy. Soc. Vict., N. S., Vol. XVI, pp. 74-78, 1908.
[18] *Op. cit.*, p. 226.
[19] Pal. Ind., Ser. XIII, Vol. I, p. 616, 1884.

Abnormal Anterior Abdominal Veins in an Indian Frog *Rana tigrina* Daud, and in an Indian Toad *Bufo melanostictus* Schneid; together with a Review of the recorded cases of Abnormalities of the Anterior Abdominal Veins.

By

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INTRODUCTION.

Ten cases of abnormal anterior abdominal veins have so far been reported from the Anuran group, nine of them belonging to the common European frog *Rana temporaria* and the remaining one to an Australian tree frog *Hyla aurea*. There is, however, no record in respect of Indian species. I would, therefore, like to place on record two further instances of abnormalities of the anterior abdominal veins, one occurring in a frog *Rana tigrina*, and the other in a toad *Bufo melanostictus*. Both the frog and the toad were mature females measuring 9 cm. in length. The abnormalities here described were met with during class demonstration in the Junior classes of this Department.

In the second part of this paper I have made an attempt to review all the recorded cases in a Table, and have grouped them in three main categories according as they show similarities with regard to persistence of embryonic features. In my conclusion I have further tried to show what bearing these abnormal cases have on the normal development of the anterior abdominal vein with special regard to its secondary connection with the hepatico-portal vein.

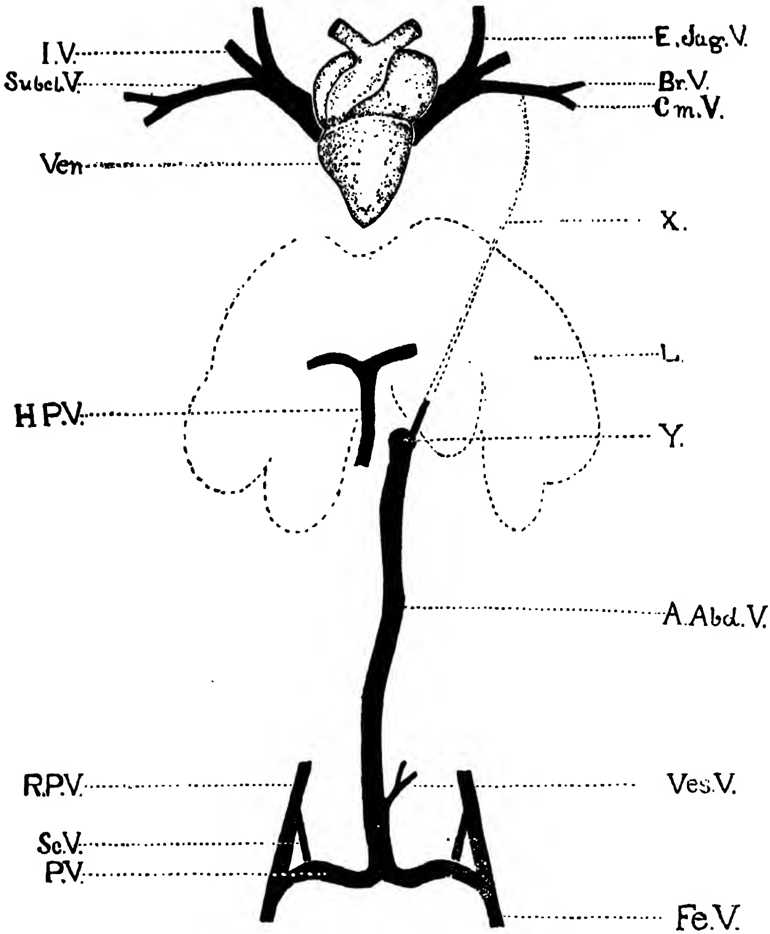
PART I.

DESCRIPTIONS OF SPECIMENS.

Specimen A. *Rana tigrina* (Text-fig. 1).

The anterior abdominal vein (A. Abd. V), in this frog, was a large vessel of greater calibre than that of a normal frog, and instead of opening anteriorly into the hepatico-portal vein (H. P. V), ended in the region of the liver (L) in the form of a knob (Y).

Text-fig. 1.

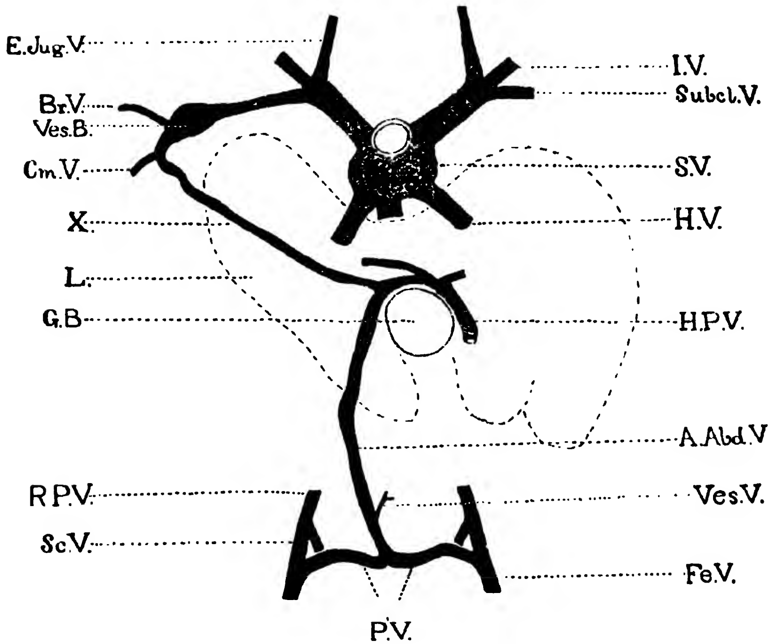


Text-fig. 1,—showing the abnormal anterior abdominal vein in *Rana tigrina* (ventral view) × 1¾.

A. Abd. V.—Anterior Abdominal Vein ; *Br. V.*—Brachial Vein ; *Cm. V.*—Musculo-cutaneous Vein ; *E. Jug. V.*—External Jugular Vein; *Fe. V.*—Femoral Vein; *H. P. V.*—Hepatico-portal Vein; *I. V.*—Innominate Vein; *L.*—Liver; *P. V.*—Pelvic Vein; *R. P. V.*—Renal Portal Vein; *Sc. V.*—Sciatic Vein; *Subcl. V.*—Subclavian Vein ; *Ven.*—Ventricle; *Ves. V.*—Vesicular Vein ; *X*—Cardiac branch of the Anterior Abdominal Vein (represented in dotted lines); *Y*—Knobbed head.

This venous knob was very thick and tough, and from its head (Y) a very fine vessel (X) was seen to proceed along the ventral body-wall towards the left side of the heart; but its course could not definitely be traced owing to a previous cut in the body-wall. From its direction, however, it may be said that it had a connection with the left subclavian vein as is generally found in similar cases

Text-fig. 2.



Text-fig. 2—showing the abnormal anterior abdominal vein in *Bufo melanostictus* (ventral view) ×1.

G. B.—Gall Bladder; *H. V.*—Hepatic Vein; *S. V.*—Sinus Venosus; *Ves. B.*—Vesicular Bulging; X—Cardiac branch of the Anterior Abdominal Vein. (Other letters as in Text-fig. 1.)

of abnormalities. I have represented this vessel, for the sake of clearness, by dotted lines (X) in the diagram. The hepatico-portal vein (H. P. V) was apparently quite normal in dividing into two usual principal afferent branches to supply the right and left lobes of the liver. Other structures were found to be normal.

The apparent truncated condition of the anterior abdominal vein has not so far been noticed in the frogs, and I, therefore, consider it fit to record this peculiarity.

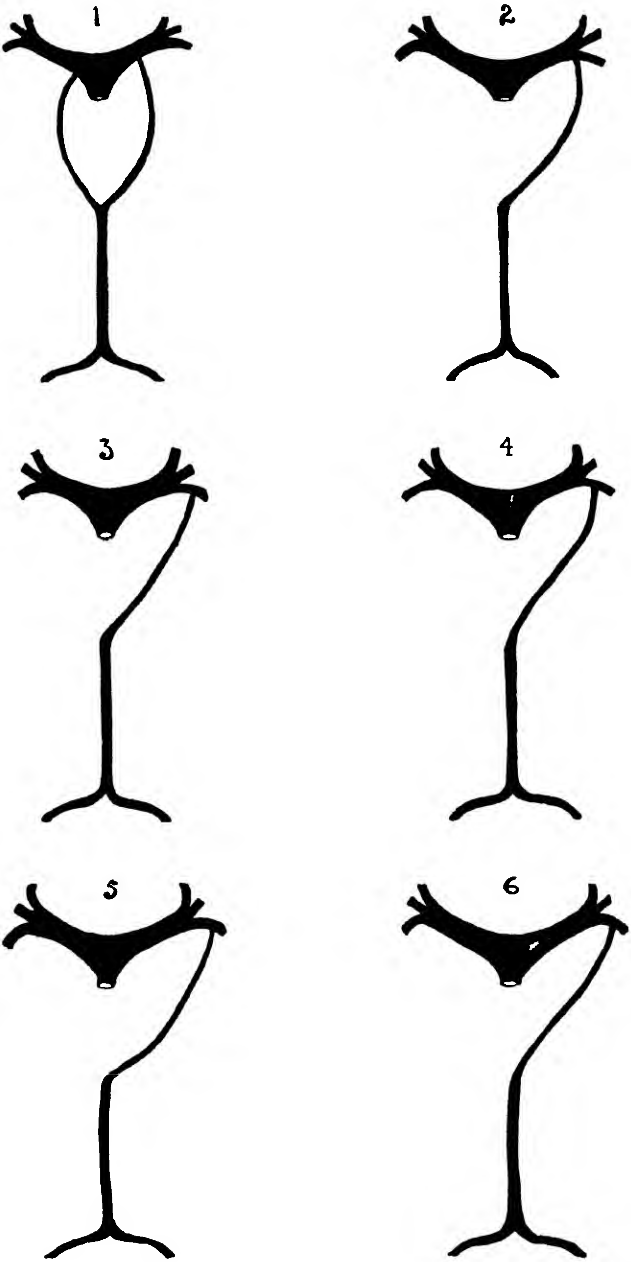
Specimen B. *Bufo melanostictus* (Text-fig. 2).

In this specimen the anterior abdominal vein (A. Abd. V) which was less prominent than in normal toads, ran right up to the liver (L) where it divided into two branches. One branch curved round the Gall Bladder (G. B) to join with the hepatico-portal vein (H. P. V), and the other ran ventrally over the right lobe of the liver to open into the subclavian vein (Subcl. V) forming a vesicular bulging (Ves. B). Curiously enough the right brachial and musculo-cutaneous veins (Br. V and Cm. V), instead of joining together to form the subclavian vein (Subcl. V), opened separately 3 mm. apart as shown in the diagram.

Abnormalities in the anterior abdominal veins are interesting in as much as they represent persistent embryonic features rather than those of reversion to piscine condition (*Ceratodus*). The latter view was pronounced by Woodland (8) and to some extent by Buller (2), but was accepted with an element of doubt by Collinge (3). The explanation of the persistence of embryonic features must be sought in the developmental history of the anterior abdominal vein. In his Vertebrate Embryology, Marshall (6) gives the following account of the development of the anterior abdominal vein in the common frog, *Rana temporaria*:

"The anterior abdominal vein is at first paired, and is in connection, not with the liver, but with the heart. The pair of vessels appear first in the ventral body wall, extending backwards a short distance from the sinus venosus; they soon extend further backwards, and acquire communications with the veins of the hind legs and of the bladder. At a later stage the two anterior abdominal veins unite at their hinder ends, in front of the bladder, while further forwards the vein of the right side disappears, the left one alone persisting. Later still, the anterior abdominal vein loses its direct communication with the sinus venosus, and acquires a secondary one with the hepatico-portal veins or afferent veins of the liver."

All the previous authors except Archer (1) and Woodland (8) have explained their respective cases in the light of the above account. The two cases, recorded by me, are likewise explicable in the same way. In the frog *Rana tigrina* the right cardiac branch of the paired embryonic abdominal veins disappeared while the left one retained its connection with the left subclavian vein. But the truncated condition of the abdominal vein cannot be accounted for in the same manner. In the toad *Bufo melanostictus*, however, the right cardiac branch persisted while the left one disappeared and its



ABNORMAL ANTERIOR ABDOMINAL VEINS

hepatico-portal connection has been secondarily acquired, presumably because of the disappearance of the left embryonic cardiac branch.

PART II.

For the purpose of facilitating review I have, in this part, appended a Table (p. 6) in which the chief peculiarities of all the previously recorded cases of abnormal anterior abdominal veins are noted, my two cases also being included. Further, as supplement to the Table, explanatory diagrams are also added in two Plates, in which, however, other types of abnormalities associated with some cases have been excluded.

Now a reference to the Table and the diagrams illustrated in Plates I and II will disclose the grouping of these recorded cases into three main categories with regard to the communications of the anterior abdominal veins with the right and left venous trunks of the heart, and also with regard to their secondary connections either with the liver or the hepatico-portal vein. The cases may be summed up, for the sake of systematising, in the following manner :

(1) The anterior abdominal vein communicates with both the *right* and *left* precavals, and has no connection either with the liver or the hepatico-portal vein.

1 case is on record :

Collinge (3), sp. No. 10.

(2) The anterior abdominal vein communicates with the *left* subclavian vein without giving off any branch to the liver or the hepatico-portal vein.

7 cases are on record :

Woodland (8).
O'Donoghue (7).
Archer (1).
Eales (4), sp. A and sp. B.
Lloyd (5).
Bhaduri, sp. A.

(3) The anterior abdominal vein communicates with the *right* venous trunks (precaval or subclavian), and always retains a connection by means of a branch either with the liver or the hepatico-portal vein.

TABLE.

Fig. No.	Authors.	Species.	Connection of Ant. Abd. vein either with liver or hepatico-portal vein.	Connection of Ant. Abd. vein either with the venous trunk of the right side or the left side of the heart.	REMARKS.
1	Collinge (3) 1915	*Rana temporaria* No. 10.	......	With *right* and *left* precavals, on the right side the vein opens very close to the sinus venosus.	
2	Archer (1) 1917	*Hyla aurea*	......	With *left* precaval at its point of termination.	
3	Woodland (8) 1906	*Rana temporaria*	......	With *left* subclavian vein	
4	Lloyd (5) 1928	*Rana temporaria*	......	With *left* subclavian vein	
5	Eales (4) 1925	*Rana temporaria* Sp. B.	......	With *left* subclavian vein	
6	,,	*Rana temporaria* Sp. A.	......	With *left* subclavian vein	Left parietal vein joins with musculo-cutaneous vein.
7	O'Donoghue (7) 1911	*Rana temporaria*	......	With *left* subclavian vein	With abnormal heart.
8	Bhaduri 1929	*Rana tigrina* Sp. A.	......	With *left* subclavian vein	Truncated abdominal vein.
9	Buller (2) 1896	*Rana temporaria*	With liver by a small branch.	With *right* precaval	
10	Eales (4) 1925	*Rana temporaria* Sp. C.	With liver.	With *right* subclavian	
11	Collinge (3) 1915	*Rana temporaria* No. 9.	With hepatico-portal vein.	With *right* precaval by a very fine branch.	
12	Bhaduri 1929	*Bufo melanostictus* Sp. B.	With hepatico-portal vein.	With *right* subclavian vein forming a vesicular bulging.	Brachial and musculo-cutaneous veins opening separately.

4 cases are on record :

Buller (2).
Collinge (3), sp. No.
Eales (4), sp. C.
Bhaduri, sp. B.

From the grouping of the cases given above, I shall try to indicate, as fairly as I can, the possible bearing these abnormalities may have on the question of the normal development of the anterior abdominal vein. The cases classed under the first two categories bear out to a large extent the two respective stages of the normal development as will be seen by a reference to Marshall's account. But it is rather curious that, in spite of the accepted explanation in terms of embryonic retention, all the four cases included under the third category do not exhibit any corresponding stage of the normal development. This lack of correspondence is due firstly to the fact, as pointed out by Buller (2), that at no time was there a stage in the normal development of the anterior abdominal vein in which the right cardiac branch persisted without the left. And secondly for the reason that there exists in these cases a secondary connection either with the liver or the hepatico-portal vein.

In connection with the second reason, however, an interesting question arises and it is this: Why should there be a secondary acquirement of connection at all in a case where the right cardiac branch has persisted without the left? It follows from the grouping of the abnormal cases, as arranged in a series in the Table and in the two Plates, that the disappearance of the *left* cardiac branch of the paired abdominal veins from the anterior venous trunks is primarily responsible for such a connection. The inference may perhaps be challenged on the ground of insufficient data, since the number of recorded cases, especially of the third category, is few. There is also a chance of this inference falling to the ground if an abnormal case, in future, be reported in which there is a left persistent abdominal vein with a secondary connection or a case of right persistent abdominal vein with no secondary connection.

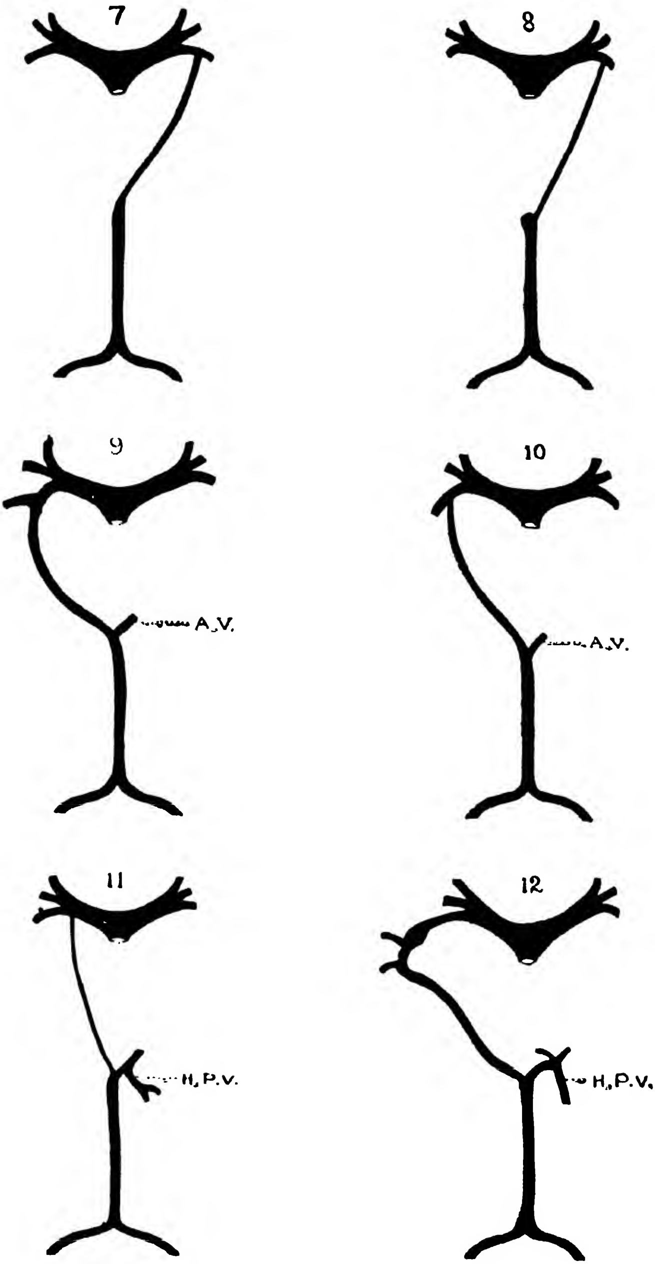
Assuming, however, the correctness of the inference that the losing of the left cardiac branch of the abdominal vein is associated with the secondary connection, it is evident that the question as to the mechanism of the acquirement of the secondary connection still

remains to be answered. It has rightly been assumed by many, though not clearly indicated by Marshall, that the acquirement of the secondary connection takes place first and then the left cardiac branch loses its communication with the sinus venosus. Such an assumption is only due to this significant fact that the right cardiac branch disappears completely first in the normal development.

Let us now see what light these abnormalities throw on this aspect of the question. In the first place we find that there is not a single case on record in which the left persisting branch is attended with a secondary connection like that found in the last phase of the normal development as described by Marshall. Such a condition may no doubt be accounted for by supposing that there is no necessity of a connection when blood can easily pass on to the heart through the left persisting abdominal vein. But real difficulty is felt in explaining the acquirement of secondary connection in the same way in cases where the right cardiac branch persists without the left. This difficulty can, no doubt to a certain measure, be obviated by accepting the view that the secondary connection is primarily due to losing of the left cardiac section from the sinus venosus. It should, however, be noted that the data derived from these abnormal cases, though not of much help in ascertaining the real mechanism of the acquirement of secondary connection, are of importance inasmuch as they serve to raise doubts about it as indicated by some previous authors.

We have already indicated elsewhere the reasons that stand in our way of explaining the four cases of the third category as true representatives of persistent embryonic features. If, however, we are inclined to consider all the recorded cases of abnormalities as persistent embryonic features, and accept the view that the different types of persistent embryonic stages in the adult condition of animals indicate corresponding stages of normal development, then a probable course of development of the anterior abdominal vein suggests itself. This course may be stated as follows : That in the normal development the right cardiac branch does not disappear *first*, but disappears only when the secondary connection has been acquired by the disappearance of the left cardiac branch from the sinus venosus.

The advantage of presuming this course of development lies in the fact that all the abnormalities can then clearly be explained as true representatives of embryonic persistence. It may be noted



ABNORMAL ANTERIOR ABDOMINAL VEINS

that the suggested course of development differs from Marshall's in the following two points :

(1) Disappearance of the right cradiac branch.
(2) Acquirement of secondary connection with the liver or the hepatico-portal vein.

In view of the fact that the course of development as suggested here by the study of abnormalities is somewhat different from Marshall's account of development, the necessity of reinvestigation of the development of the anterior abdominal vein arises. Moreover, it may be observed that since Marshall's account was published the development of the anterior abdominal vein has not received any further attention. Hochstetter,[1] who has done so much on the development of the vascular system of the vertebrates, has practically accepted Marshall's generalisation following Goette.[2]

Finally, I wish to express my sincere thanks to my colleague Mr. D. Mukerji, Lecturer of the Calcutta University, for kind encouragement and suggestions.

[1] Hochstetter, F.,

1888...Beiträge zur vergleichenden Anatomie und Entwicklungsgeschichte des Venensystem der Amphibien und Fische. *Morph. Jahrb., Leipsig*, p. 169.

[2] Goette, A.,

1875 ..Entwicklungsgeschichte der Unke. *Leipzig.*

References.

1. Archer, E.,
1917. ....Abnormal circulation of a frog. *Melbourne Proc. R. Soc., Victoria*, XXXIII, pp. 96-97.

2. Buller, A. H. R.,
1896......Abnormal Anterior Abdominal Vein in a Frog. *Journ. Anat. & Physiol., London*, XXX, pp. 211-214.

3. Collinge, W. R.,
1915......Some Abnormal Developments in the Vascular System of the Frog. *Journ. Anat. & Physiol., London*, L, pp. 37-42.

4. Eales, N. B.,
1925......Three Cases of Abnormal Anterior Abdominal Veins in the Frog. '*Nature*,' *London*, 115, No. 2895, p. 606.

5. Lloyd, J. H.,
1928......On Abnormalities in *Rana temporaria* chiefly affecting the Vascular System. *Proc. Zool. Soc., London*, pp. 307-315.

6. Marshall, A M.,
1893......Vertebrate Embryology, London, p. 184.

7. O'Donoghue, C. H.,
1911......Two Cases of Abnormal Hearts and one of an Abnormal Anterior Abdominal Vein in the Frog. *Zool. Anz., Leipzig*, XXXVII, 37, pp. 35-38.

8. Woodland, W. N. F.,
1910......An Abnormal Anterior Abdominal Vein in the Frog. *Zool. Anz., Leipzig*, XXXV, pp. 626-627.

Explanation of Plate I.

Fig. 1.—Collinge's Case, No. 10.
Fig. 2.—Archer's Case.
Fig. 3.—Woodland's Case.
Fig. 4.—Lloyd's Case.
Fig. 5.—Eales's Case, Sp. B.
ig. 6.—Eales's Case, Sp. A.

(For details see Table.)

Explanation of Plate II.

Fig. 7.—O'Donoghue's Case.
Fig. 8.—Bhaduri's Case, Sp. A.
Fig. 9.—Buller's Case.
Fig. 10.—Eales's Case, Sp. C.
Fig. 11.—Collinge's Case, No. 9.
Fig. 12.—Bhaduri's Case, Sp. B.

A.V.—Afferent vein from the anterior abdominal to the liver; *H.P.V.*—Hepatic-portal vein.

(For details see Table.)

Notes on Some Rocks of the Rajmahal Hills.

BY

P. C. DATTA, M.Sc.

(With plates.)

The present note is the result of the study of rock specimens undertaken at the suggestion of Prof. Dasgupta. They were collected by him from the part of the Rajmahal Hills lying in the immediate neighbourhood of Simra Bungalow and known locally as the Gandhesari Hill.

The earliest notice of these rocks was by Dr. Buchanan [1] according to whom the hill, *i.e.*, the Gandhesari Hill which is made up of these rocks, marks the position of an old crater. This statement was later on supported by Mr. Ball [2] who described the rock as a pinkish trachyte, porphyritic at some places. No microscopic or chemical study of these rocks was, however, made by him. Subsequently Col. MacMahon studied these rocks under the microscope [3] and found them to be andesite and not trachyte. The specimens supplied to me were also of a pinkish colour as noted by the previous observers, but this colour is likely due to the decomposition of the rock. A fresh surface shows dark brown colour and considerable pittings. A dark-coloured ferromagnesian mineral can be made out, but no specific determination with the lens is possible as the rock is of a very fine-grained nature. Kaolin and ochre are present. The specific gravity of the specimens was found to range from 2·48—2·80, the average being 2·66.

Several sections of the comparatively less decomposed rocks were prepared. Under the microscope the rock is found to be porphyritic with a holo-crystalline groundmass resembling what is generally found to be characteristic of a dolerite. The section consists of

[1] *Gleanings in Science*, Vol. 2, pp. 5-8, 38-39, 1831.

[2] *Memoirs, Geological Survey of India*, Vol. 13, Pt. 2, p. 66, 1877.

[3] *Records, Geological Survey of India*, Vol. 20, pp. 106-107, 1887.

numerous lath-shaped plagioclase felspar crystals showing their characteristic lamellar twinning. The field is full of numerous haematitic patches which are due to the decomposition of augite which is often found forming the cores of these patches. Lemon-yellow coloured isotropic patches resembling Palagonite, a substance already recorded from the basalts of the Rajmahal Hills,[1] were also detected under the microscope.

A chemical analysis of the rock gave the following result:

SiO_2	... ...	50·00
Al_2O_3	... ...	15·10
Fe_2O_3	... ...	6·16
FeO	... ...	5·41
MgO	... ...	6·43
CaO	... ...	11·34
Na_2O	... ...	4·30
K_2O	... ...	0·08
$H_2O\pm$	... ...	1·59
TiO_2	... ...	n.d.
P_2O_5	... ..	n.d.
		100·41

From this analysis the following norm has been calculated:

Orthoclase	0·50
Albite	31·96
Nepheline	2·27
Magnetite	9·05
Anorthite	21·40
Diopside	27·00
Olivine	6·42
	98·60
H_2O	1·59
	100·19

[1] *Records, Geol. Surv. of India*, Vol. 22, pp. 226-234, 1889.

Fig. 1
(In plain Polarised Light)
Magnification × 40.

Fig. 2

(Under Crossed Nicols)

Magnification × 40.

From the norm given above the position of the rock has been determined as follows:

Class	...	... Salfemane
Order	...	... Perfelic
Rang	...	... Alkali-calcic
Subrang	...	... Persodic

Symbol—III, 5, 3, 5.

The present analysis compares favourably with the analysis of the Deccan trap basalts as published by Washington.[1] A peculiar feature of the present rock, however, is its low percentage of iron-oxides amounting to 11·57 while in the case of the Deccan trap basalts the iron-oxides vary from 12·6—14·5 with an average of 13·1. In the present case the ferrous oxide does not dominate over the ferric as in the case of the Deccan trap basalts, but they are almost equal in amount. The amount of K_2O is very small as was expected from a microscope examination which shows little or no Orthoclase felspar. The CaO in the majority of the Deccao trap basalts varies from 9·49—11·26 and in the present case it is 11·34. Thus it is clear from a careful study that the rock is neither a trachyte as suggested by Buchanan and Ball nor an andesite as opined by MacMahon, but a basalt of the usual plateau type.

In conclusion I wish to acknowledge with thanks the help I received from Mr. N. N. Chatterjee while carrying on my work and also Prof. Dasgupta for suggesting the work to me.

Explanation of plate.

Fig. 1. Section of Gandhesari basalt (with nicols parallel).
Fig. 2. Section of the same (with nicols crossed).

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[1] *Bulletin, Geological Society of America*, Vol. 33, p. 774, 1922.

ARTIFICIAL CULTURE OF **Ganoderma colossus** IN ERLENMEYER FLASK.

FIG. 1.

Tissue-culture of Ganoderma colossus, Fr.

BY

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Ganoderma colossus is a saprophyte growing usually at the bases of dead trunks. I have collected it often from the bases of dead mango (*Mangifera indica*) and *Ficus religiosa* trunks. I have fully described it with two plates in our College Bulletin No. II (of the Carmichael Medical College—1921).

For purposes of this culture the specimen was collected from the lower part of the trunk of a dead and standing *Ficus religiosa* tree on the roadside in front of the Belgachia Government Veterinary College in July, 1928.

A fresh sporophore was selected for the culture-work, and a small piece of tissue was taken aseptically from the context of the sporophore, and put in sterilised malt-extract agar-tubes (malt-extract 3%, Agar 2%, water 100 c.c.). In the course of a few days a white floccose cottony growth on the upper surface of the slant was noticed in some of the tubes; immedietely after, a mycelial fragment was transferred successively from one set of tubes to the other; in this way chances of contamination were avoided.

In the course of about ten days, when it was found that the whole of the slants of the malt-extract agar-tubes was covered with pure white mycelial growths, a number of sterile flasks containing the sterilised bread-medium were inoculated with a subculture from the pure growth. The flasks with pure growths were kept within a glass-chamber at the ordinary room temperature, and the fungus continued to grow favourably for a number of months. In all these flasks the white hyphae changed to yellow colour after a fortnight, and then gradually turned to brownish colour on the border with yellowish patches here and there at the centre. In the course of two months there was noticed a copious exudation of drops of coloured liquid from the upper surface of the media, and gradually the

whole upper surface put on a loose powdery appearance giving rise to a large number of double-walled spores, and the regular porous surface was finally formed on the edges of the upper surface of the 'media in contact with the glass-walls of the flasks (Fig. 1) after an interval of about three months from the date of inoculation ; these are the abortive sporophores formed in artificial cultures of *Polypores* as has been remarked by Dow V. Baxter (Michigan Academy of Science, Arts and Letters, Vol. IV, pp. 63-64—1924) ; typical pilei are hardly formed in such cultures. The porous surface showed a number of pores of varying sizes, the pores were exuling drops of glistening liquid, and the pore-tubes did not show any basidia but only a number of peculiar double-walled spores resembling the basidio-spores typical of the genus *Ganoderma.* The lower surface of the media on their edges finally acquired the actual laccate deep-yellow coloration of the upper surface of the original specimen (*Ganoderma colossus*) in patches. One interesting variation regarding these peculiar dobule-walled spores was noticed that in culture they were mostly round and very few oval, whereas in nature they were mostly oval and very few round.

Mycelial fragments were transferred aseptically from one of the flasks to blocks of sterilised wood of *Mangifera indica and Spondias mangifera* within sterilised Roux-tubes. The Roux-tubes were kept within a glass-chamber at the ordinary room-temperature. First of all, there was a white vegetative growth almost covering the external surface of the wood-blocks. Gradually, the growth slowed down, and there was formation of yellowish patches here and there and also on the border, which ultimately turned brownish. The edges of the wood-blocks put on a loose powdery appearance showing the formation of small pores in resupinate patches (Fig. 2) ; this was subsequently followed by the appearance of a small protrusion close to the top of the wood-block (Fig. 2) after an interval of about five and a half months from the date of subculture ; this is the abnormal fructification formed in culture without any pilear formation. It showed a number of pores (resembling the pores of the original specimen) within which the peculiar double-walled spores of *Ganoderma colossus* were found. The spores were mostly round. Both the wood-blocks gave rise to similar tiny fructifications, and externally they show no sign of decay.

Artificial culture of **Ganoderma colossus** within Roux tube.

Fig. 2.

Dynamics of the Pianoforte String and Hammer

By

Panchanon Das, M.Sc.

FOREWORD

A theory of the dynamics of pianoforte string and hammer was worked out by the author[1] and published in a series of papers during the past few years. Its conclusions have received experimental support from the investigations of Datta,[2] George,[3] and George and Beckett.[4] References to the theory appear in the volume Acoustics of the Handbuch der Physik (Julius Springer, Berlin) in the articles Nos. 36 and 37 by Prof. C. V. Raman and in § 24, 25, 26 and 27 by Prof. A. Kalähne. It appeared to the author that as the work was published in a series of disconnected articles, it would be useful to bring them together in a connected account of the subject. Incidentally some minor corrections and some new material have been incorporated in the present Memoir.

As Rayleigh[5] has given a general method of studying the vibrations of strings when the forces producing them are known as a function of time, the problem of the motion of the pianoforte string appears at first sight to present no features of new interest to the investigator. But the force, here exerted by the hammer, is not known *a priori*, so that early workers [6] used to assume it as a known function of time. The determination of the pressure of impact of the hammer on the string is thus the key to the dynamics of the

[1] P. Das, Proc. Ind. Assoc. Cult. Sc., Vol. VII, 1921; Vol. IX, 1926; Vol. X, 1926; Ind. Jour. Phys., Vol. I, 1927; Proc. Lond. Phys. Soc., Vol. 40, 1927; Phil. Mag., Vol. 5, 1928. These papers will be referred to hereafter as Nos. 1, 2, 3, 4, 5 and 6.

[2] Datta, Proc. Ind. Assoc. Cult. Sc., Vol. VIII, p. 107.

[3] George, Proc. Roy. Soc., 108, p. 284, 1925.

[4] George, and Beckett, Proc. Roy. Soc., 111, p. 111, 1927. In this connection see author's paper No. 4, where George and Beckett's experimental results were explained by his own theory.

[5] Treatise on Sound, Vol. I, p. 187 Ed. 2nd.

[6] Helmholtz, Sensation of Tone. Delemer, Ann. Soc. Scient. de Bruxelles, Tome 30, 1905-6.

pianoforte. In the case where the string is struck at a point very near a fixed end of the same, Kaufmann [1] first successfully derived an expression for the pressure, by regarding the shorter of the two parts into which the string is divided by the point struck, as remaining straight during the impact. Following in the footsteps of Love [2] in his treatment of St. Venant's problem of the vibration of a rod struck at one end, *viz.*, by dividing the time of impact into a series of equal intervals, the present author derived exact solutions of Kaufmann's problem, when the vibrations of the shorter length could not be neglected.

While the author's method was based on successive functional solutions, Professor C. V. Raman and Dr. B. Banerji [3] solved the problem somewhat earlier by the method of trigonometric series. It will be shown later on that their theory and ours lead to practically identical results.

Of the hard hammer, which has been discussed in the Section I of this Memoir, the principal features investigated are (1) the pressure of impact, (2) the duration of the same, (3) the impulse and energy communicated by the hammer, (4) the displacement of the string, (5) the amplitude of overtones, and (6) the reflections from both ends of the string.

In Section II, where the yielding hammer is dealt with, the influence of the elasticity of the hammer on the tone-quality has been studied with the help of exact solutions.

[1] Wiedemann's Annalen, Bd. 54, S. 675, 1895.

[2] Treatise on Elasticity, p. 412 Ed. Second.

[3] Raman and Banerji, Proc. Roy. Soc., Vol. 97, p. 108 (1920).

SECTION I.

HARD HAMMER.

§ 1. *Pressure of Impact.*

We regard the pianoforte problem as reduced to that of a string stretched between two fixed points A and B and struck at some point O by a rigid unyielding hammer of pointed shape having an initial velocity V and mass m. Take O (Fig. 1) as the origin of axes and the x-axis

Fig. 1

along OA, and the y-axis in the plane of vibration. If $OA=a$, and $OB=b$, the co-ordinates of A and B are (a, o) and $(-b, o)$ respectively.

Let the displacement of any point (x, o) of the string be denoted by

$$\left.\begin{array}{r} y_2 \text{ for } x>0 \\ y_1 \text{ for } x<0 \\ \text{and } y_0 \text{ for } x=0 \end{array}\right\}.$$

The equation of motion of the hammer during the impact is

$$m\frac{d^2y_0}{dt^2}=T\left(-\frac{\partial y_1}{\partial x}+\frac{\partial y_2}{\partial x}\right)_{x=0} \quad \text{.. (1)}$$

where T is the tension of the string.

The equation of motion of the string is

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2} \quad \text{... (2)}$$

where y stands for y_2 or y_1 according as $x>0$ or <0.

In the region $x \geq 0$, let the solution of (2) be of the form

$$y_2 = F(ct+x) + f(ct-x).$$

The end A being fixed, the value of y_2 is zero for $x=a$. Hence,

$$F(ct+a) + f(ct-a) = 0.$$

Therefore $F(z) = -f(z-2a)$ and we may write

$$y_2 = f(ct-x) - f(ct+x-2a). \qquad \ldots \quad (3)$$

Since the velocity and inclination of the string to the x-axis are initially zero at all points in the region $0 \leq x \leq a$, we have

$$\left(\frac{\partial y_2}{\partial t}\right)_{t=0} = f'(-x) - f'(x-2a) = 0,$$

and $$\left(\frac{\partial y_2}{\partial x}\right) = -f'(-x) - f'(x-2a) = 0.$$

Hence $f'(-x) = f'(x-2a) = 0$ for $0 < x < a$.

In other words $f'(z) = 0$ for $0 > z > -2a$.

Therefore $f(z)$ is a constant during the same interval and we may equate the constant to zero. Hence the value of y_2 is simply $f(ct-x)$ from the instant $t=0$ to the instant $t=(2a-x)/c$, since $-f(ct+x-2a)$ is zero in that interval.

Thus, when $0 < t < (2a-x)/c$,

$$y_2 = f(ct-x). \qquad \ldots \quad (4)$$

In particular, when $0 < t < 2a/c$,

$$y_0 = f(ct). \qquad \ldots \quad (5)$$

Now in an actual pianoforte, the length a is not more than (1/7)th of the total length $l = a + b$, and the duration of impact is usually about half the fundamental period of the string. Thus the impact generally terminates before the reflection from the end B distant b from O reaches the hammer at O. Hence so far as the hammer is concerned the reflected wave from the end B may be left out of consideration. Thus between O and B, *i.e.*, in the region $x \leq 0$, we have simply,

$$y_1=\phi(ct+x).$$

The continuity at the origin gives

$$y_0=(y_1)_{x=0}=(y_2)_{x=0},$$

$$\text{or } \phi(ct)=f(ct).$$

Therefore $y_1=f(ct+x)$ ... (6)

In order to find y_0 or $f\,(ct)$ in the aforesaid intervals $0<ct<2a$, we substitute (4), (5) and (6) in the equation (1) and get

$$f''(ct)+kf'(ct)=0$$

$$\text{where } k=2\text{T}/mc^2.$$

If ρ be the mass per unit length of the string, then

$$c^2=\text{T}/\rho, \text{ so that } k=2\rho/m. \qquad \text{... (7)}$$

The first integral of the foregoing differential equation is

$\frac{1}{c}\cdot\frac{dy_0}{dt}=f'(ct)=\text{A}e^{-kct}$, where A is an arbitrary constant.

Since the hammer is rigid, we may take the initial value of $\frac{dy_0}{dt}$ to be the same as that of the hammer, *viz.*, V.

Thus $\text{A}=\text{V}/c$; and in the interval $0<ct<2a$, we have

$$f'(ct)=\frac{\text{V}}{c}\,e^{-kct}. \qquad \text{... (8)}$$

When $t>(2a-x)/c$, so that $ct+x-2a$ is positive, the function $-f(ct+x-2a)$ is no longer zero so that we then have

$$y_2=f(ct-x)-f(ct+x-2a). \qquad \text{... (9)}$$

In particular, $y_0=f(ct)-f(ct-2a)$. ... (10)

The condition of continuity at the origin gives

$$y_1=f(ct+x)-f(ct+x-2a). \qquad \text{... (11)}$$

If $2a<ct<4a$, the argument $ct-2a$ lies between O and $2a$, hence we have from (8)

$$f'(ct-2a)=\frac{V}{c}e^{-k(ct-2a)}. \qquad \ldots \quad (12)$$

In order to find the form of $f(ct)$ in the interval $2a<ct<4a$, we substitute (9), (10) and (11) in the equation (1) and get

$$f''(ct)+kf'(ct)=f''(ct-2a) \qquad \ldots \quad (13)$$

Since $2a>ct-2a>0, f''(ct-2a)$ is derived by differentiating (12). Thus :—

$$f''(ct)+kf'(ct)=-\frac{V}{c}e^{-k(ct-2a)}$$

Integrating this we get

$$f'(ct)=Ae^{-kct}-\frac{V}{c}e^{-k(ct-2a)}\,k(ct-2a) \qquad \ldots \quad (14)$$

The fact that the hammer cannot have a discontinuous change of velocity owing to its finite mass renders $\frac{dy_0}{dt}$ continuous at $ct=2a$, when the reflection from the end A reaches the hammer. In other words

$$\mathrm{Lt}_{ct-2a=+0}\left\{f'(ct)-f'(ct-2a)\right\}$$

$$=\mathrm{Lt}_{ct-2a=-0}\left\{f'(ct)-f'(ct-2a)\right\}, \qquad \ldots \quad (15)$$

the first limit being approached from the side $ct-2a>0$, and the second from the side $ct-2a<0$.

Now in the interval $4a>ct>2a, f'(ct)$ is given by (14) and $f'(ct$ $2a)$ is given by (11). Hence the expression on the left in (15) is

$$Ae^{-2ak}-\frac{V}{c}.$$

Again in the interval $2a>ct>0, f'(ct)$ is given by (8) and $f'(ct-2a)$ is zero. Hence the expression on the right in (15) is

$$\frac{V}{-} e^{-2ak}$$

Equating these we get $A=\frac{V}{-}(e^{2ak}+1)$.

Therefore we have from (14),

$$f'(ct)=\frac{V}{c}e^{-kct}+\frac{V}{c}e^{-k(ct-2a)}\{1-k(ct-2a)\} \quad \ldots \quad (16)$$

in the interval $4a>ct>2a$.

When $6a>ct>4a$, the function $f''(ct-2a)$ will be derived from (16) and the arbitrary constant in $f'(ct)$ obtained by integrating (13) will be determined by a condition similar to (15). In this manner we find that in the interval $6a>ct>4a$,

$$f'(ct)=\frac{V}{c}e^{-kct}+\frac{V}{c}e^{-k(ct-2a)}\{1-k(ct-2a)\}$$

$$+\frac{V}{c}e^{-k(ct-4a)}\left\{1-2k(ct-4a)+\frac{k^2}{\underline{|2}}(ct-4a)^2\right\} \quad \ldots \quad (17)$$

Similarly when $8a>ct>6a$, the expression for $f'(ct)$ is obtained by adding to (17) the following term :—

$$\frac{V}{c}e^{-k(ct-6a)}\left\{1-3k(ct-6a)+\frac{3k^2}{\underline{|2}}(ct-6a)^2-\frac{k^3}{\underline{|3}}(ct-6a)^3\right\}.$$

Generally in the interval $(n+1)\,2a>ct>n.2a$, it can be shown that

$$f'(ct)=\sum_{r=1}^{r=n+1} f'_r(ct) \quad \ldots \quad (18)$$

$$\text{where } f'_{r+1}(z)=\frac{V}{c}e^{-k(z-r.2a)}\left\{1-rC_1.k(z-r.2a)\right.$$

$$+{}^rC_2.\frac{k^2}{\lfloor 2}(z-r.2a)^2-{}^rC_3.\frac{k^3}{\lfloor 3}(z-r.2a)^3+\ldots$$

$$\left.\ldots(-1)^r\frac{k^r}{\lfloor r}(z-r,2a)^r\right\} \quad \ldots \quad (19)$$

so that we have,

$$\left\{\begin{array}{l} f'_1(z)=\dfrac{V}{c}e^{-kz} \\ f'_2(z)=\dfrac{V}{c}e^{-k(z-2a)}\{1-k(z-2a)\} \\ f'_3(z)=\dfrac{V}{c}e^{-k(z-4a)}\left\{1-2k(z-4a)+\dfrac{k^2}{\lfloor 2}(z-4a)^2\right\}\text{etc.} \end{array}\right. \quad (20)$$

Thus in the interval $2a>ct>0, f'(ct)=f'_1(ct)$.

" " " $4a>ct>2a, f'(ct)=f'_1(ct)+f'_1(ct)$.

" " " $6a>ct>4a, f'(ct)=f'_1(ct)+f'_2(ct)+f'_3(ct)$,

and so on, where f'_1, f'_2, f'_3 etc, are given by (20).

In the interval $(n+1)2a>ct>n.2a$, it is easily seen that

$$\frac{1}{c}\cdot\frac{dy_0}{dt}=f'(ct)-f'(ct-2a)$$

$$\sum_{r=1}^{r=n+1}f'_r(ct)-\sum_{r=1}^{r=n}f_r'(ct-2a), \quad \ldots \quad (21)$$

the function f'_r being given by (19).

Thus it is obvious that if we divide the time of contact between the hammer and the string into a number of intervals or epochs, each of magnitude $2\,a/c$, the motion of the hammer is different in different epochs as is obvious from the change of form of $f'\ (ct)$. The result (20) and (21) were obtained by the author in a different manner (Paper No. 1, p. 18).

On differentiating (21) with respect to t and making use of (7) and (20) we find that in the $(n+1)$th epoch defined by $(n+1)\ 2a \geq ct > n.2a$, the acceleration of the hammer is given by

$$\frac{d^2y_0}{dt^2} = -2\rho Vc\Big[e^{-kt} + e^{-k(ct-2a)}\ \{1-k(ct-2a)\}$$

$$+e^{-k(ct-4a)}\ \Big\{1-2k(ct-4a)+\frac{k^2}{\lfloor 2}(cr-4a)^2\Big\}+\ldots$$

$$+e^{-k(ct-n.2a)}\ \Big\{1-n.k(ct-n.2a)+\frac{n(n-1)}{\lfloor 2}.\frac{k^2}{\lfloor 2}(ct-n.2a)^2+..,$$

$$..,+(-1)^n\frac{k^n}{\lfloor n}(ct-n.2a)^n\Big\}\Big] \qquad \ldots \quad (22)$$

The pressure exerted by the hammer on the string is obtained by multiplying by m the acceleratoin y_0'' given by (22). If we denote by F_1, F_2, F_3, etc., the values of this pressure of impact during the first, second, and third, etc., epochs respectively, then we find on disregarding the negative sign that

$$\begin{cases} F_1 = 2\rho Vce^{-kct} \\ F_2 = 2\rho Vc\Big[e^{-kct} + e^{-k(ct-2a)}\Big\{1-k(ct-2a)\Big\}\Big] \\ F_3 = 2\rho Vc\Big[e^{-kct} + e^{-k(ct-2a)}\Big\{1-k(ct-2a)\Big\} \\ \qquad + e^{-k(ct-4a)}\Big\{1-2k(ct-4a)+\frac{k^2}{\lfloor 2}(ct-4a)^2\Big\}\Big] \\ \text{etc.} \end{cases} \qquad (23)$$

By putting $t=0$ in F_1 as given by (23) we see that the initial value of the pressure is $2\rho Vc$, so that the pressure does not grow gradually from zero but is finite from the beginning. At the instant $ct=2a$, the pressure as calculated from F_1 is $2\rho Vce^{-2ak}$, but its value calculated from F_2 is $2\rho Vce^{-2ak}+2\rho Vc$. Thus there is a discontinuous increase of pressure of magnitude $2\rho Vc$ at the instant $ct=2a$. It is obvious that the same thing occurs at each of the instants $ct=4a$,

$6a$, and so on. The existence of these discontinuous pressure variations were first established on the principle of conservation of momentum by Raman and Banerji (*loc. cit.*).

In order to study the course of the development of the pressure of impact with time, curves were drawn, with time as abscissa, and pressure as ordinate. Such a pressure-time curve is given in Fig. 2. For comparison, the corresponding kind of curve plotted on the basis of Raman-Banerji's theory has been taken from their work and is reproduced in Fig. 3.

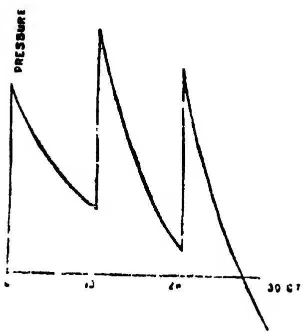


Fig. 2

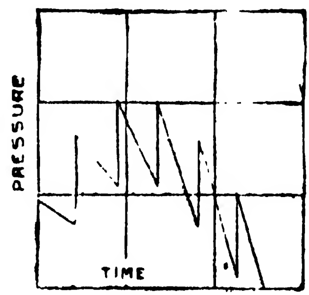


FIG. 3

W. H. George (*loc. cit.*) has experimentally investigated these curves by means of a Dudell Oscillograph and his diagrams, of which one is reproduced in Fig. 4, points out these discontinuous pressure variations remarkably. But George's curves were probably complicated by reflections from the other end B, so that the resemblance is not quite complete.

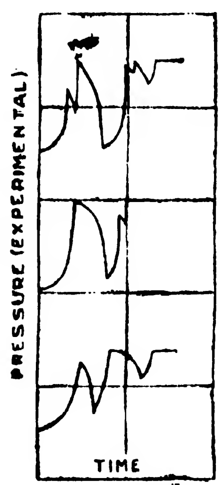


FIG. 4

A comparison of our theory with Kaufmann's shows that while the latter gives a rough indication of the course of development of pressure with time, the former points out the fine-structure of these pressure-time curves. If we draw a mean curve between the upper and lower extremities of the vertical lines periodically occurring in the curves of Figs. 2 and 3, the curve so obtained (not shown here) is found to be closely given by Kaufmann's formula for pressure which runs as follows :—

$$F = \frac{V}{c\eta} e^{\xi ct} \sin\left(\eta ct + 2\tan^{-1}\frac{\eta}{\xi} \right) \dots\dots\dots \quad \dots \quad (24)$$

where $\xi = -\frac{\rho}{2m}$ and $\eta = \sqrt{\frac{\rho}{ma} - \frac{\rho^2}{4m^2}}$, the notation being ours.

The equation of a zig-zag of broken lines, having amplitude ρVc and period $2a/c$ has the Fourier form

$$F = \frac{\rho Vc}{\pi} \sum_{1}^{\infty} \frac{1}{n} \cdot \sin\frac{n\pi ct}{a} \quad \dots \quad (25)$$

If we add (24) and (25) so as to superimpose the zig-zag on Kaufmann's curve we get a close approximation to the result of our rigorous analysis. The dotted curve in Fig. 5 is plotted by adding (24) and (25), while the continuous curve is plotted with the help of our exact formulae (23). It is obvious that (25) arises out of the vibrations of the length a, which was neglected by Kaufmann.

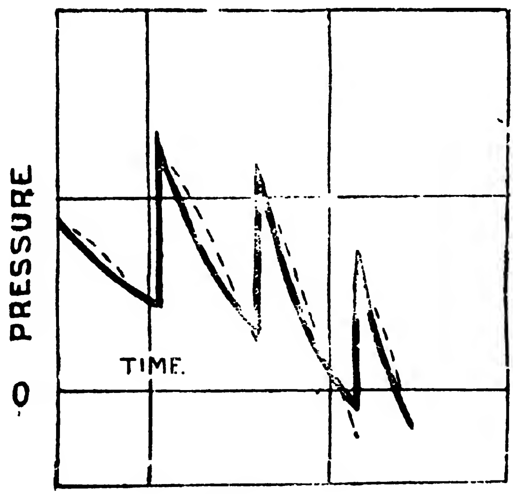


FIG. 5

§ 2. *Duration of Impact.*

As long as the hammer is in contact with the string, it exerts the pressure F of which the form has been studied in the previous article. It is reasonable to suppose that the hammer parts company with the string when the pressure F becomes zero. The duration of impact is thus the smallest positive root of the equation $F=0$.

An algebraic expression for this duration of impact can be derived only in special cases, since the equation $F=0$ is of the n^{th} degree in t. It is obvious at a glance that the impact cannot terminate in the first epoch since the root of $F_1=0$ is infinite. The equation $F_2=0$ has a finite root, *viz.*,

$$t_1 = \frac{2a}{c} + \frac{1+e^{2ak}}{ck}$$

which gives the duration of impact, if it terminates in the second epoch.

The condition that the impact may not extend into the third epoch is that $t_1 < 4a/c$, or $1+e^{-2ak} < 2ak$. This inequality is satisfied if $2ak$ is somewhat greater than unity. Since $k=2\rho/m$, the foregoing condition is equivalent to the length a being somewhat greater than $4\rho/m$. This fact is of practical value to experimental workers.

As the string length a is made smaller and smaller the reflections from the fixed end A arrive in quicker succession, and the impact extends into higher and higher epochs, although the actual magnitude of the duration t_1 may not increase. In fact t_1 diminishes with a, as will be shown shortly.

We have seen that the pressure-time curve drawn with Kaufmann's formula is a good approximation to ours. Hence an approximate value of the duration of contact may at first be calculated by means of Kaufmann's formula, and the order number of the epoch in which the impact terminates may be determined by it. If we now apply Newton's formula for approximation to the roots of an eqation, a fairly accurate value of the duration of impact may

be obtained. Newton's method was applied by the author in paper No. 3, § 5, for drawing the curve of the duration of contact against

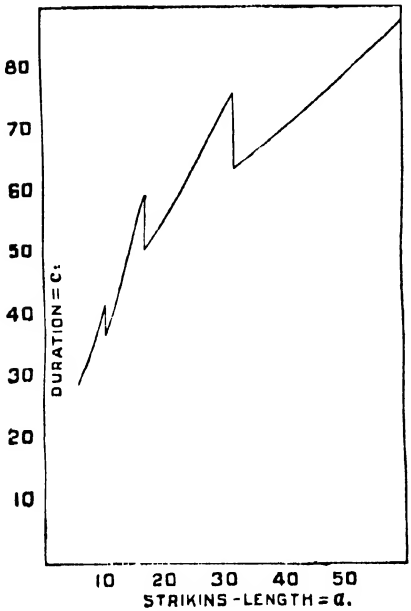


Fig. 6

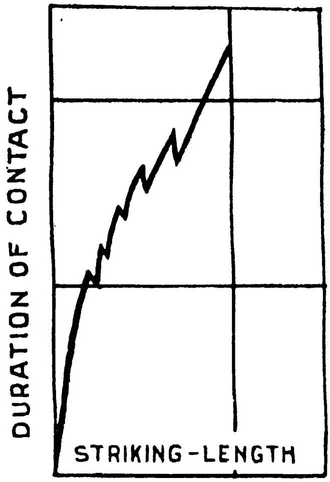


Fig. 7

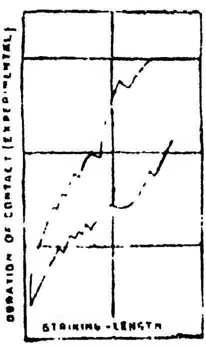


Fig. 8

the striking length a. The same curve has been reproduced here in Fig. 6. It will be noted that although the average slope of the curve is towards the origin, which means that t_1 diminishes with a, yet there are discontinuous changes in the value of t_1 from point to point. Raman-Banerji's (*loc. cit.*) theoretical and experimental curves of t_1 against a are given in Figs. 7 and 8 respectively. A

comparison of Fig. 8 with Figs. 6 and 7 shows that both the theories are borne out by experiments.

The graphic method however is universally applicable and simplifies the study of t_1. The duration t_1 is obviously the intercept on the t-axis made by the pressure-time curve. This method was actually adopted by Datta (*loc. cit.*), who undertook to test the author's theory by measuring the amplitudes of the harmonics of the struck string. Our formula for t_1 has recently been verified by Ganguly & Banerji.[1]

Although we have assumed in the foregoing paragraphs that the hammer falls off the string once for all, the moment the pressure of the hammer becomes equal to zero, recent experiments by George and Beckett[2] have shown however that very often the string overtakes the receding hammer and again establishes contact with the same. These repeated contacts however cannot be investigated with exactness on the basis of our theory, although a roughly approximate treatment of the same has been actually given by the author in paper No. 4.

§ 3. *Impulse and Energy.*

Recently some experimenters[3] have investigated the dynamics of the pianoforte string by experimentally determining the momentum and the energy communicated to the string by hammer. Hence a theoretical treatment of this aspect of the subject may be useful.

The momentum imparted by the hammer is $\int_0^{t_1} F dt$, where F is the pressure given by (23) and t_1 is the duration of impact. Obviously it is graphically represented by the area of the curve of F against t between the origin and the point where F-t curve cuts the t-axis. If we divide the momentum measured graphically as above by the mass of the hammer, we get its change of velocity. Hence if the initial velocity of the hammer is known, its final velocity as well as its loss of energy can be calculated.

While the graphic method just described is applicable in all cases, a general analytical expression for the loss of energy cannot be derived since it involves the duration of contact.

[1] Ganguly & Banerji. Phil. Mag. Feb. 1929.

[2] George and Beckett, Proc. Roy. Soc., 114, p. 111, 1927.

[3] Ditto.

With the help of (21) and (20) of § 1, it can however be easily proved that the velocity v of the hammer at any instant t is given as a fraction of the initial velocity V in the second, third and fourth epochs respectively by

$$\left.\begin{aligned}
&\text{(II Epoch)}, \ \frac{v}{V} = e^{-kct} - k(ct-2a)e^{-k(ct-2a)} \\
&\text{(III \ ,, \ )}, \ \frac{v}{V} = \text{Ditto} + e^{-k(ct-4a)}\left\{-k(ct-4a) + \frac{k^2}{\underline{|2}}(ct-4a)^2\right\} \\
&\text{(IV \ ,, \ )}, \ \frac{v}{V} = \text{Ditto} + \text{Ditto} + e^{-k(ct-6a)} \\
&\left\{-k(ct-6a) + k^2(ct-6a)^2 - \frac{k^3}{\underline{|3}}(ct-6a)^3\right\}
\end{aligned}\right\} \quad \dots \quad (26)$$

If we substitute the duration of impact t_1 for t in the expressions (26) for v, we get the final velocity of the hammer at the instant it falls off the string. The energy lost by the hammer then equals $\frac{1}{2}m(V^2 - v^2)$.

George and Beckett[1] have experimentally measured the loss of energy for a number of cases and plotted $(V^2-v^2)/V^2$ against $a/(a+b)$, and their curves have been reproduced in Figs. 9, 10, 11 and 12. From the numerical data supplied by the above-mentioned authors the value of $(V^2-v^2)/V^2$ was calculated in terms of $a/(a+b)$ with the help of the formula (26) and curves were plotted by the present writer in paper No. 4.

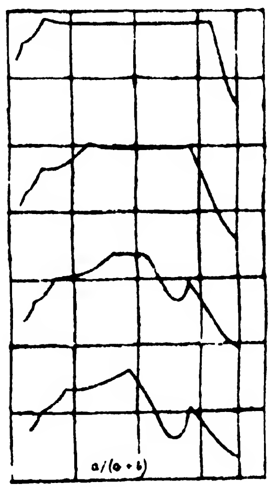


Fig. 9, 10, 11 & 12

They are reproduced in Figs. 13, 14, 15 and 16 which correspond to the experimental Figs. 9, 10, 11 and 12, respectively. It is to be noted that not only the general course of the experimental curves have been satisfactorily explained by the theory,

[1] loc. cit. Foreword.

but in the simpler cases there has been almost exact numerical

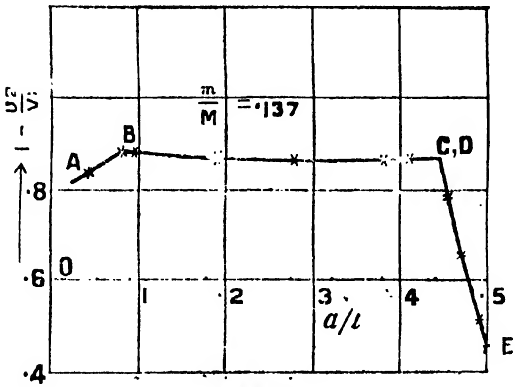


Fig. 13

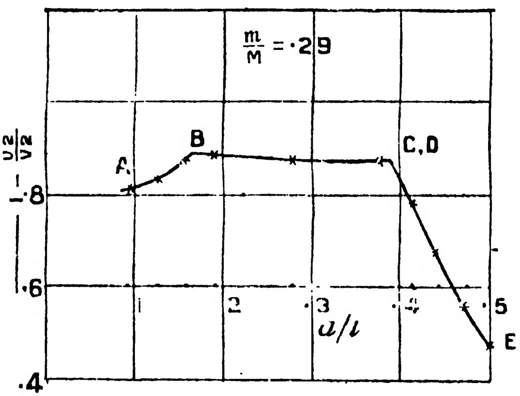


Fig. 14

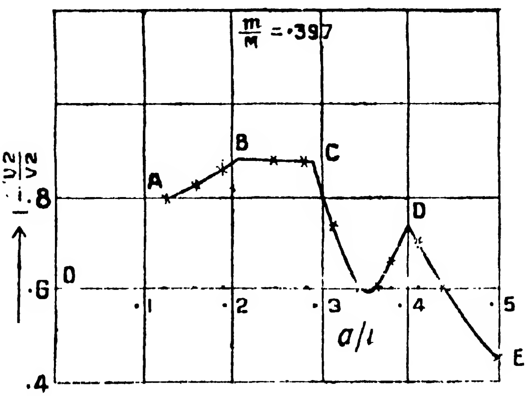


Fig. 15

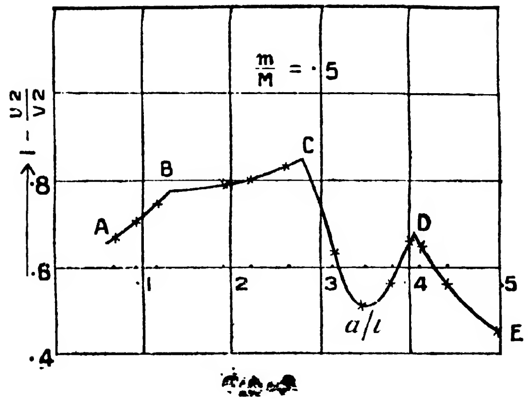


Fig. 16

agreement. This has so far been the best experimental verification of our theory.

The part towards the extreme right of the curves in the foregoing figures, where the ratio $a/(a+b)$ is nearly $\frac{1}{2}$ has been affected by reflections from the remote end B. The formulæ (32), (33) and (37) of § 6 where reflections from both ends of the string have been considered in particular cases, have been used to calculate this extreme part of the curves.

§ 4. *Displacement.*

As yet we have derived only the first integral of the equation of motion (1) which is of the second order. The second integral of the same, which gives the displacements of the hammer and different parts of the string can be easily obtained.

It was shown in paper No. 1, which can also be followed up from § 1 of the present Memoir, that the displacement y_o can be expressed in terms of the integrals of the function $f_r'(z)$ given by (20). For example,

in the I Epoch, $y_o = f_1(ct)$,

in the II Epoch, $y_o = f_1(ct) + f_2(ct) - f_1(ct-2a)$,

in the III Epoch, $y_o = f_1(ct) + f_2(ct) + f_3(ct) - f_1(ct-2a) - f_2(ct-2a)$,

and so on.

The physical interpretation is that the first pulse $f_1(ct)$ generated at the instant $t=0$ is reflected at the end $x=a$, and the reflection $-f_1(ct-2a)$ reaches the hammer at $x=0$ at the instant $t=2a/c$, when a new pulse $f_2(ct)$ arises. Thus $f_2(ct)$ again, is reflected at $x=a$ and reaches the hammer at the instant $t=4a/c$, when another new pulse $f_3(ct)$ is brought into existence. Thus a pulse $f_{n+1}(ct-x)$ contributes to the displacement of a point $(x, 0)$ only from the instant $ct-x=n.2a$ and not before that.

As regards the actual forms of f_{n+1} it was shown in paper No. 3 [result (45)] that

$$f_{n+1}(ct)=-\frac{V}{kc}\, e^{-kz_n}\left[-kz_n+\sum_{2}^{n} {}^{n-1}C_{n-1}\, \frac{(-k)^r z^r_n}{\lfloor r}\right] \quad \ldots \quad (27)$$

where $z_n=ct-n.2a$.

In particular, we have

$$\begin{cases} f_1(ct)=-\dfrac{V}{kc}\left(e^{-kct}-1\right) \\ f_2(ct)=\dfrac{V}{kc}\, e^{-k(ct-2a)}\, k(ct-2a) \\ f_3(ct)=-\dfrac{V}{kc}\, e^{-k(ct-4a)}\Big\{-k(ct-4a) \\ \qquad\qquad +\dfrac{k^2}{\lfloor 2}(ct-4a)^2\Big\}. \end{cases} \quad (28)$$

When the impact is over, it was shown in § 3, paper No. 3, that no more new pulses are generated, but those already in existence only suffer successive reflections. It was also shown there, that if t_1 were the duration of impact, then the value of $f_{r+1}(ct)$ would be constant and equal $f_{r+1}(ct_1)$ at any time $t>t_1$.

Thus the two most important properties of the function $f_{r+1}(ct)$ are :—

$$\left.\begin{aligned} &f_{r+1}(ct)=0, \text{ when } t>r.2a/c, \\ &\text{and } f_{r+1}(ct)=f_{r+1}(ct_1), \text{ when } t>t_1 \end{aligned}\right\} \quad (29)$$

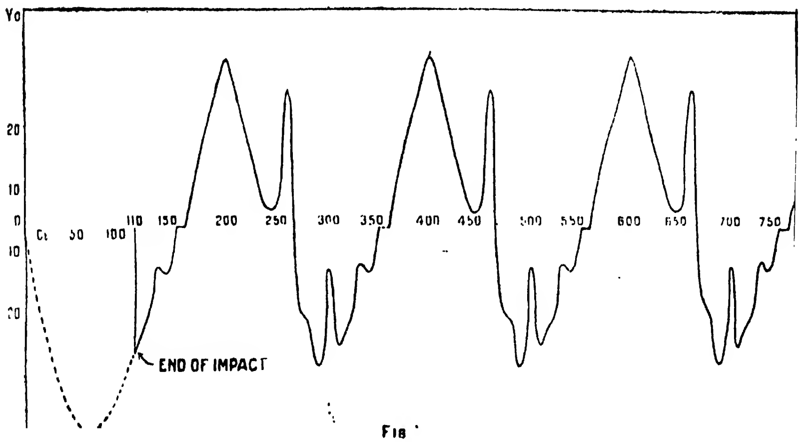
When t lies between t_1 and $r.2a/c$, $f_{r+1}(ct)$ is given by (27).

Hence the rule for calculating the displacement at any point on the string is as follows :—

First prepare a table of all the functional waves brought into existence and their successive reflections as shown below :—

$$f_r(ct-x), -f_r(ct+x-2a), f_{r+1}(ct-x-2a-2b),$$
$$-f_{r+1}(ct+x-4a-2b), \text{ etc.}$$

Next select some instant $t>t_1$, and examine the argument of each of the above pulses. If the argument is greater than ct_1, write ct_1 for the argument. If the argument is less than $r.2a$, reject the function f_r. The sum of all the functions so selected gives the total displacement at the instant t in question.



STRUCK AND OBSERVED AT ONE FIFTH

Fig. 17

A numerical example has been worked out in § 3 of paper No. 3, and Fig. 17 is a reproduction of the displacement time diagram of a point on the string in a particular case discussed there. The study of this diagram is extremely interesting from the dynamical point of view as it shows how the transition from an aperiodic motion to a periodic one takes place. The dotted part of the curve corresponds to the motion during the impact, and is seen to be aperiodic. The periodic motion begins the moment the hammer falls off.

§ 5. *Amplitude of Harmonics.*

The amplitude of harmonics is determined by the form of the normal co-ordinates of the vibrating string. If ϕ_s be the normal co-ordinate of order s, it is given by

$$\phi_s = \frac{2}{n\rho l}\int^{t} \sin n(t-t')\Phi_s dt',$$

where ρ=mass per unit length of the string, l=length of the same, $n=s\pi c/l$ and Φs is the generalised component of the force, *i.e.*,

$$\Phi s = \mathrm{F}\ \sin\frac{s\pi a}{l},\ \mathrm{F}\text{ being the pressure of impact given by (23).}$$

Thus,

$$\phi_s = \frac{2}{s\pi c\rho}\ \sin\frac{s\pi a}{l}\int_{0}^{t} \sin n(t-t')\mathrm{F}.dt' \qquad \ldots \quad (30)$$

The most practical method of evaluating the integral in (30) is by the use of graphs. One has only to plot F against t and multiply the ordinates by sin nt' and cos nt' and measure the area of curves so obtained, between $t=0$ and $t=t_1$.

Datta (*loc. cit.*) experimentally verified the author's theory of the pianoforte and used the graphic method of calculation. The tables Nos. I and II taken from his paper show the calculated and observed values side by side for values of the striking length $a=(1/10)$th and and (1/9)th of the whole length l of the string respectively. The agreement is remarkably close.

The integral (30) can however be analytically evaluated in simple cases. For example, if the impact terminates in the second epoch, the upper limit of the integral is

$$t_1 = \frac{2a}{c} + \frac{1+e^{2ak}}{ck}$$

Thus,

$$\phi_s = \frac{2}{n\rho l} \sin \frac{s\pi a}{l} \int_0^{2a/c} \sin n(t-t')\mathrm{F}_1 . dt'$$

$$+ \frac{2}{n\rho l} \sin \frac{s\pi a}{l} \int_{2a/c}^{t_1} \sin n(t-t')\mathrm{F}_2 . dt'$$

where F_1 and F_2 are given by (23).

By actually carrying out the integration and making use of the fact that t_1 is the root of $\mathrm{F}_2=0$, it can be shown that

$$\phi s \Big/ \frac{4\mathrm{V}}{nl} \sin \frac{s\pi a}{l} = a_s \sin nt + b_s \cos nt,$$

TABLE I ($a=l/10$).

Harmonics.	Observed amplitude.	Calculated amplitude.
Fundamental ...	·3700 cm.	·3975 cm.
Octave	·0588 ,,	·0599 ,,
Third	·0133 ,,	·0133 ,,
Fourth	·0130 ,,	·0131 ,,
Fifth	·0078 ,,	·0079 ,,
Sixth	·0100 ,,	·0147 ,,
Seventh	·0020 ,,	·0022 ,,
Eighth	·0157 ,,	·0173 ,,
Ninth	·00112 ,,	·00128 ,,

TABLE II ($a=l/9$).

Harmonics.	Observed amplitude.	Calculated amplitude.
Fundamental ...	·4240 cm.	·4322 cm.
Octave	·0345 ,,	·0365 ,,
Third	·0167 ,,	·0173 ,,
Fourth	·0217 ,,	·0240 ,,
Fifth	·0046 ,,	·0043 ,,
Sixth	·0029 ,,	·0027 ,,
Seventh	·0071 ,,	·0068 ,,
Eighth	·0020 ,,	·0023 ,,
Tenth	·0022 ,,	·0025 ,,

where

$$\left.\begin{matrix} as \\ bs \end{matrix}\right\} = \frac{\begin{matrix}\cos\\ \sin\end{matrix}\left(-\theta\right)+\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{2an}{c}-\theta\right)}{\sqrt{k^2c^2+n^2}}$$

$$+\frac{kc}{k^2c^2+n^2}\left[e^{-1-e}\quad\begin{matrix}\cos\\ \sin\end{matrix}\left(nt_1-2\theta\right)-\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{2an}{c}-2\theta\right)\right] \qquad \ldots \quad (31)$$

It is obvious from (31) that the lengths a and l do not occur there as a mere ratio but exist independently. Thus the ratio a/l is not the sole determinant of tone-quality. This fact which is generally overlooked by experimenters, accounts for their conflicting general conclusions.

A more general discussion is reserved for the other Section.

§ 6. *Reflections from both Ends.*

The foregoing theory of the pianoforte labours under the restriction that one extremity of the string is so remote from the point struck by the hammer that the impact terminates before the reflections from this extremity reach the hammer. This limitation is easily removed in a few special cases, as will be shown here. A more general but approximate solution was given by the author in paper No. 2, § 4. It was shown there that if P_a stands for the pressure due to reflections from the end A only and P_b the pressure corresponding to the end B only, the actual pressures of impact is approximately the sum $P_a + P_b$.

It was shown in § 1 that the moment the hammer strikes the string, a pulse $f_1(ct \pm x)$ starts both ways. It was also shown in § 4 that when the pulse f_1 $(ct-x)$ reaches the near end A, it is reflected in the form $-f_1$ $(ct+x-2a)$, which on reaching the hammer gives rise to the pulse f_2 $(ct \pm x)$. Now since f_2 (ct) is a function of the distance a, this dependence of f_2 on a should be rendered explicit in order to avoid confusion, since we are now going to consider reflections from the other end also.

Thus throughout this article, let us write f_2 $(ct \pm x-2a)$ in the place of f_2 $(ct \pm x)$, so that we have from (28)

$$f_2\ (ct-2a)=\frac{V}{kc}e^{-k\ (ct-2a)}\ k\ (ct-2a).$$

Similarly the wave generated at the instant the pulse $-f_1$ $(ct-x-2b)$ from the end B reaches the hammer, is f_2 $(ct-2b)$

$$=\frac{V}{kc}e^{-k\ (ct-2b)}\ k\ (ct-2b).$$

Thus in the new notation the displacement $y_2=f_1\ (ct)+f_2\ (ct)+$ $-f_1$ $(ct-2a)$ in the interval $2a<ct<4a$.

If the string is struck nearly at the middle point so that b is only slightly greater than a, the pulse $-f_1(ct+x-2b)$ will overtake the hammer at the instant $ct=2b$ and give rise to the pulse f_2 $(ct \pm x-2b)$. Hence in the interval $2b<ct<4a$, we have

$$\begin{aligned} y_0 &= f_1\ (ct)+f_2\ (ct-2a)-f_1\ (ct-2a) \\ &\quad +f_2\ (ct-2b)-f_1\ (ct-2b) \qquad \ldots \quad (32) \end{aligned}$$

At the instant $ct=4a$, the pulse $f_2\ (ct-2a)$ will reach the hammer back again as $-f_2\ (ct-4a)$ and give rise to $f_3\ (ct-4a)$. Thus in the interval $4a \leq ct \leq 2a+2b$,

$$
\begin{aligned}
y_0 = f_1\ (ct) + f_2\ (ct-2a) - f_1\ (ct-2a) \\
+ f_3\ (ct-4a) - f_2\ (ct-4a) \\
+ f_2\ (ct-2b) - f_1\ (ct-2b).
\end{aligned}
$$

At the instant $ct=2a+2b$, the pulses $-f_1\ (ct-2a)$ and $-f_1\ (ct-2b)$ will undergo reflection from the ends B and A respectively as $f_1\ (ct-2a-2b)$ and $f_1\ (ct-2b-2a)$. Each of these will give rise to a pulse $-f_2\ (ct-2a-2b)$.

Similarly $f_2\ (ct-2b)$ and $f_2\ (ct-2a)$ will be reflected from A and B as $-f_2\ (ct-2b-2a)$ and $-f_2\ (ct-2a-2b)$ respectively, each of which will give rise to the pulse $f_3\ (ct-2a-2b)$. Thus in the interval $2a+2b<ct<4b$, we have

$$
\begin{aligned}
y_0 = f_1\ (ct) + f_2\ (ct-2a) - f_1\ (ct-2a) \\
+ f_3\ (ct-4a) - f_2\ (ct-4a) \\
+ f_2\ (ct-2b) - f_1\ (ct-2b) \\
+ 2\ f_1\ (ct-2a-2b) - 2\ f_2\ (ct-2a-2b) \\
- 2\ f_2\ (ct-2a-2b) + 2\ f_3\ (ct-2a-2b) \quad \ldots \quad (33)
\end{aligned}
$$

At the instant $ct=4b$, the following pulse will be added to the foregoing value of y_0 :—

$$
f_3\ (ct-4b) - f_2\ (ct-4b) \qquad \ldots \quad (34)
$$

Thus in the interval $4b<ct<4b+2a$, y_0 is the sum of (33) and (34). In this manner it is possible to put down the value of the displacement at any instant.

If we put $b=a$, we get the case of a string struck at the middle point, investigated by Kaufmann. Thus in the interval $2a \leq ct \leq 4a$, we have from (32) :—

$$
y_0 = f_1\ (ct) + 2f_2\ (ct-2a\ - 2f_1\ (ct-2a).
$$

Making use of (28) and re-arranging the terms we get:—

$$y_0=\frac{V}{c}\left[e^{-kct}\left\{2e^{2ak}\left(ct-2a+\frac{1}{k}\right)-\frac{1}{k}\right\}-\frac{1}{k}\right] \quad \dots \quad (35)$$

If we write $K=2Sg/mc^2$, the result (35) is found to be identical with the expression (20) of Kaufmann's paper. Kaufmann's formula has been experimentally verified by W. H. George,[1] so that this agreement is another proof of the sound dynamical basis of our method.

If the impact in this case extends beyond the instant $ct-4a$, we have from (33) and (34),

$$\begin{aligned} y_0 &= f_1(ct)+2f_2(ct-2a)-2f_1(ct-2a) \\ &\quad +4f_3(ct-4a)-6f_2(ct-4a)+2f_1(ct-4a) \\ &= \frac{V}{kc}\left[1-e^{-kct}+2e^{-k(ct-2a)}\left\{1+k(ct-2a)\right\}\right. \\ &\quad \left.-2e^{-k(ct-4a)}\left\{1+k(ct-4a)+k^2(ct-4a)^2\right\}\right] \quad \dots \quad (36) \end{aligned}$$

Kaufmann did not carry the investigation beyond the second epoch It can be shown that the value of y_0 in the third epoch as derived by following Kaufmann's method is the same as is given by (36).

To take another example, let the string be struck at one-third, so that b nearly equals $2a$; let $b<2a$. Then one can easily establish the following results :—

In the interval $2a<ct<4a$,

$$y_0=f_1(ct)+f_2(ct)-f_1(ct-2a).$$

In the interval $4a\leq ct\leq 4b$,

$$\begin{aligned} y_0 &= f_1(ct)+f_2(ct-2a)-f_1(ct-2a) \\ &\quad +f_3(ct-4a)-f_2(ct-4a). \end{aligned}$$

In the interval $2b<ct>2b+2a$,

$$\begin{aligned} y_0 &= f_1(ct)+f_2(ct-2a)-f_1(ct-2a) \\ &\quad +f_3(ct-4a)-f_2(ct-4a) \\ &\quad +f_2(ct-2b)-f_1(ct-2b) \quad \dots \quad (37) \end{aligned}$$

[1] George, Phil. Mag., Vol. 48, pp. 34 and 48, 1924.

If we differentiate y_0 twice with respect to t and multiply it by m, we get the pressure of impact. The formulæ (11) and (13) of paper No. 4, were derived in this way from (32) and (37) respectively. The loss of energy calculated with the help of these formulæ was in good agreement with George and Beckett's experimental values. Thus our extended theory embracing reflections from both ends has also been borne out by experiments.

SECTION II.

Elastic Hammer.

§ 1. *Exact Theory.*

The investigations of the previous section are based on the assumption that the hammer is made of rigid inelastic material. But in an actual pianoforte the hammer-core is covered with felt, which has elastic resilience and yields considerably under pressure. In this section we shall give a short account of our investigations (papers Nos. 2 and 3) that were extended to the case of an yielding or elastic hammer. For the sake of mathematical simplicity we suppose that the compression of the hammer under the pressure of impact is governed by Hooke's law.

We regard the elastic hammer as having a pointed shape, so that the area of contact with the string is small. If u be the compression of the felt-covering and ζ be the displacement of the rigid core of the hammer, then the following kinematic relation between u, ζ and y_0 holds good :—

$$\zeta = y_0 + u \qquad \ldots \quad (38)$$

the quantities, y_0, y_1, y_2, etc., having the same significance as before.

If F be the pressure between the hammer and the string, the equation of motion of the former is

$$F = m\frac{d^2\zeta}{dt^2} = T\left(-\frac{\partial y_1}{\partial x} + \frac{\partial y_2}{\partial x}\right)_{x=0} \qquad \ldots \quad (39)$$

Now according to Hooke's law, F is proportional to the compression u; let it equal Eu. Making use of this and the relation (38) in (39) we get

$$m\frac{d^2y_0}{dt^2} + m\frac{d^2u}{dt^2} = T\left(-\frac{\partial y_1}{\partial x} + \frac{\partial y_2}{\partial x}\right)_{x=0} = -Eu \qquad \ldots \quad (40)$$

During the first epoch ($0 < ct < 2a$), the solution of (2) takes the functional form

$$\begin{cases} y_1 = f_1(ct+x), \text{ for } x<0 \\ y_2 = f_1(ct-x), \text{ for } x>0 \\ y_0 = f_1(ct), \qquad \text{for } x=0. \end{cases}$$

Substituting these in (40) we get

$$mc^2 f_1''(ct) + m\frac{d^2u}{dt^2} = -2\mathrm{T} f_1'(ct) = -\mathrm{E}u \qquad \dots \quad (41)$$

which gives

$$u = \frac{2\mathrm{T}}{\mathrm{E}} f_1'(ct) = \lambda f_1'(ct) \qquad \dots \quad (42)$$

where

$$\lambda = \frac{2\mathrm{T}}{\mathrm{E}} \qquad \dots \quad (43)$$

Substituting the value of u from (4z) in (41) we get

$$f_1'''(ct) + \frac{1}{\lambda} f_1''(ct) + \frac{\mathrm{E}}{mc^2} f_1'(ct) = 0.$$

The solution of this differential equation is of the form

$$f_1'(ct) = \mathrm{A}e^{\alpha ct} + \mathrm{B}e^{\beta ct} \qquad \dots \quad (44)$$

where A and B are constants of integration and α, β are the roots of the equation

$$x^2 + \frac{x}{\lambda} + \frac{\mathrm{E}}{mc^2} = 0 \qquad \dots \quad (45)$$

Now just at the instant the impact begins, the compression u is zero. Hence from (42) and (44), we get

$$\mathrm{A} + \mathrm{B} = 0, \text{ or } \mathrm{A} = -\mathrm{B} \qquad \dots \quad (46)$$

Again, if the impinging velocity of the hammer is **V**, then

$$\left(\frac{d\zeta}{dt}\right)_{t=0} = \mathrm{V}.$$

Hence from (38), we get

$$V=\left(\frac{dy_0}{dt}+\frac{du}{dt}\right)_{t=0}$$

But $y_0=f_1\,(ct)$ and $u=\lambda f_1'\,(ct)$. Hence from (44) and (46) we get

$$V=cf_1'\,(o)+\lambda cf_1''\,(o)$$

$$=\lambda cA\,(\alpha-\beta).$$

$$\text{or } A=-B=\frac{V}{\lambda c(\alpha-\beta)} \quad \ldots \quad (47)$$

Thus the pressure of impact during the epoch is seen from (42) and (47) to be given by

$$F_1=Eu=2Tf_1'\,(ct)$$

$$=\frac{2TV}{c\lambda(\alpha-\beta)}\left\{e^{\alpha ct}-e^{\beta ct}\right\} \quad \ldots \quad (48)$$

The pulse $f_1(ct-x)$ becomes reflected in the form $-f_1(ct+x-2a)$ and overtakes the hammer again at the instant $ct=2a$, when the second epoch begins. At the same time a new disturbance $f_2(ct\pm x)$ arises. Thus during the second epoch we have

$$\begin{cases} y_1=f_1\,(ct+x)+f_2\,(ct+x)-f_1\,(ct+x-2a) \\ y_2=f_1\,(ct-x)+f_2\,(ct-x)-f_1\,(ct+x-2a) \\ y_0=f_1\,(ct)+f_2\,(ct)-f_1\,(ct-2a) \end{cases} \quad \ldots \quad (49)$$

Proceeding as before one finds that $f_2\,(ct)$ satisfies the following differential equation :—

$$f_2'''\,(ct)+\frac{1}{\lambda}f_2''\,(ct)+\frac{E}{mc^2}\,f_2'\,(ct)=\frac{1}{\lambda}f_1''\,(ct-2a).$$

On solving this equation and determining the constants from the condition of continuity of velocities of the string and the hammer at the instant $ct=2a$, we find that

$$f_2'(ct)=\frac{V}{c}\cdot\frac{1}{\lambda^3(a-\beta)^3}\left[e^{a(ct-2a)}\left\{1+a\lambda(a-\beta)(ct-2a)\right\}\right.$$

$$\left.-e^{\beta(ct-2a)}\left\{1+\beta\lambda(\beta-a)(ct-2a)\right\}\right] \quad \dots \quad (50)$$

Substituting (49) in (40) we get the pressure F_2 in the second epoch,

$$F_2=Eu=2T[f_1'(ct)+f_2'(ct)] \quad \dots \quad (51)$$

and we have already found the form of $f_1'(ct)$ and $f_2'(ct)$.

It has been shown in the same manner in §1 of paper No. 3, that the $f_3(ct\pm x)$ that arises at the third epoch is given by

$$f_3'(ct)=e^{a(ct-2a)}\left\{a_0+a_1(ct-4a)+a_2(ct-4a)^2\right\}$$

$$+e^{\beta(ct-2a)}\left\{b_0+b_1(ct-4a)+b_2(ct-4a)^2\right\} \quad \dots \quad (52)$$

$$\left.\begin{aligned} &\text{where } a_0=-b_0=\frac{V}{c}\cdot\frac{1+2a\beta\lambda^2}{\lambda^5(a-\beta)^5};\\ &a_1=\frac{V}{c}\cdot\frac{a(1-\beta\lambda)}{\lambda^4(a-\beta)^4};\quad a_2=\frac{V}{c}\cdot\frac{1}{\lambda^3(a-\beta)^3}\cdot\frac{a^2}{\lfloor 2}\end{aligned}\right\} \quad \dots \quad (53)$$

and b_1, b_2 are derived by interchanging a, β in the expressions for a_1 and a_2 given in (53).

General solutions for any epoch have been given in the paper cited.

If the hammer is very hard, but not absolutely rigid, the foregoing formulæ for pressure become somewhat simplified. It is easily seen that the quantities E/T and ρ/m have the same dimension, *viz.*, L^{-1}. The condition that the hammer is very hard is mathematically expressed by taking E/T as very large compared with ρ/m. In that case the roots a, β respectively become $-2\rho/m$ and $-E/2T$,

approximately. A little calculation (paper No. 2, pp. 308-310) at once shows that the pressures of impact during the first three epochs become :

$$F_1 = 2\rho Vc\,[e^{\alpha ct} - e^{\beta ct}] \qquad \dots \quad (54)$$

$$F_2 = 2\rho Vc[e^{\alpha ct} + e^{\alpha(ct-2a)}\{1+\alpha(ct-2a)\}$$
$$- e^{\beta ct} - e^{\beta(ct-2a)}\{1-\beta(ct-2a)\}] \qquad \dots \quad (55)$$

$$F_3 = 2\rho Vc[e^{\alpha ct} + e^{\alpha(ct-2a)}\{1+\alpha(ct-2a)\}$$
$$+ e^{\alpha(ct-4a)}\{1+2\alpha(ct-4a)+\frac{\alpha^2}{\underline{|2}}(ct-4a)^2\}$$
$$- e^{\beta ct} - e^{\beta(ct-2a)}\{1-\beta(ct-2a)\}$$
$$- e^{\beta(ct-4a)}\{1-\beta(ct-4a)+\frac{\beta^2}{\underline{|2}}(ct-4a)^2\}] \qquad (56)$$

We can pass on to the limiting case of the absolutely rigid hammer by taking E to be infinitely large. It is well-known that the limiting value of $x^n e^{-x}$ is zero when $x \longrightarrow \infty$, n being finite. This at once shows that the terms involving β in (54), (55) and (56) drop out, so that the values of F_1, F_2, F_3 become identical with those given in (23) for the rigid hammer, it being remembered that $\alpha = -k = -2\rho/m$. Thus the problem of the rigid hammer is a special case of the more general problem of the elastic hammer.

§ 2. *Approximate Theory.*

The complexity of the solutions derived so far lies in the fact their forms are different in different epochs. If the striking-length a is so small as to admit of its being regarded as a straight rod turning about a hinge, just as in Kaufmann's treatment, this multiple character of the solutions disappears and a great simplification arises. The problem of the elastic hammer was first viewed in this simplified aspect by Bhargab and Ghosh.[1]

[1] Bhargab & Ghosh, Phil. Mag., Vols. 47 and 49, 1926.

The displacement of the string instead of being of the form given in § 1, will now be

$$\left.\begin{aligned} y_0 &= f(ct+x) \\ y_2 &= \frac{a-x}{a} y_0 \\ y_0 &= f(ct) \end{aligned}\right\} \quad \dots \quad (57)$$

The equation of motion is still given by (40) except that m has to be replaced by $m_0 = m + \rho a/3$, after Kaufmann. The compression u is then given by

$$u = \frac{T}{E}\left(\frac{T_0}{a} + \frac{dy_0}{dt}\right) \quad \dots \quad (58)$$

Substituting (38) and (58) in (40) we get

$$\frac{m_0 T}{Ec} \cdot \frac{d^3 y_0}{dt^3} + m_0\left(1 + \frac{T}{aE}\right)\frac{d^2 y_0}{dt^2} + \frac{T}{c} \cdot \frac{dy_0}{dt} + \frac{T}{a} y_0 = 0 \quad \dots \quad (59)$$

Before integrating (59) Bhargab and Ghosh[1] dropped the term involving $\frac{d^3 y_0}{dt^3}$ on the ground that its co-efficient was a small quantity of the order 10^{-3} in actual pianofortes. But that is no reason why the corresponding arbitrary constant of integration also shall be of the same order of magnitude. Their formula leads to the dynamically unwarrantable conclusion that the initial velocity of the string and the initial pressure of impact are both finite for the elastic hammer.

The complete integral of (59) is (paper No. 4).

$$y_0 = Ae^{qct} + Be^{\xi ct} \sin(\eta ct + \theta)$$

where A, B, θ are arbitrary constants and q and $\xi \pm i\eta$ are the roots of the equation

$$x^3 + \left(\frac{E}{T} + \frac{1}{a} + \frac{\rho a}{3m_0}\right)x^2 + \frac{\rho}{m_0} \cdot \frac{E}{T}\left(x + \frac{1}{a}\right) \quad \dots \quad (60)$$

[1] Bhargab & Ghosh, Lon. Phys. Soc. Proc., Vol. 40, p. 227, 1921.

From the initial conditions that the velocity and displacement of the string at the origin are each zero and the velocity of the hammer is V, we get

$$\left.\begin{aligned} A&=\frac{V}{c}\cdot\frac{E}{T}\cdot\frac{1}{\eta^2+(q-\xi)^2} \\ B&=\frac{V}{c}\cdot\frac{E}{T\eta}\cdot\frac{1}{\sqrt{\eta^2+(q-\xi)^2}} \\ \theta&=\tan^{-1}\frac{\eta}{q-\xi} \end{aligned}\right\} \quad \ldots \quad (61)$$

If ρ/m is small compared with the largest root of the equation (60) a further simplification arises. The roots are then approximately given by :—

$$\left.\begin{aligned} q&=-\left(\frac{E}{T}+\frac{1}{a}\right) \text{ and } \xi\pm i\eta, \\ \text{where} \quad \xi&=-\frac{aE}{aE+T}\cdot\frac{\rho}{2m_0} \\ \text{and} \quad \eta&=\sqrt{-\frac{2\xi}{a}-\xi^2} \end{aligned}\right\} \quad \ldots \quad (62)$$

The pressure is then given by

$$F=Eu=2T\left(\frac{y_0}{a}+\frac{dy_0}{dt}\right)$$

$$=\frac{V}{c}\cdot\frac{E^2}{Tq^2}\left(e^{\xi ct}\cos\eta\,ct-e^{qct}\right)-\frac{V}{c}\cdot\frac{E}{qa\eta}\,e^{\xi ct}\sin\eta\,ct \quad \ldots \quad (63)$$

which may be written in the form

$$F=A_1e^{qct}+B_1e^{\xi ct}\sin(\eta\,ct+\theta_1) \quad \ldots \quad (64)$$

It is obvious at a glance that if we put $t=0$ in (63), the pressure F vanishes, as it ought to. Again, if we put $E=\infty$, (63) becomes identical with Kaufmann's formula for rigid hammer.

§ 3. *Comparison with Hard Hammer.*

Before discussing the influence of the elasticity of the hammer on the quality of the pianoforte tone, we shall first examine the purely dynamical effect of replacing a hard hammer with an yielding or elastic specimen. The study of the pressure-time curve for the cases

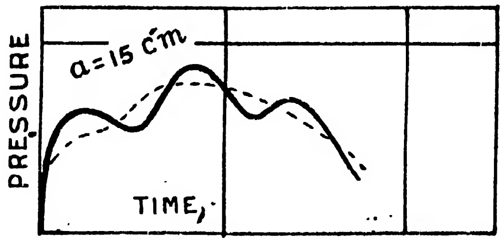


FIG. 18

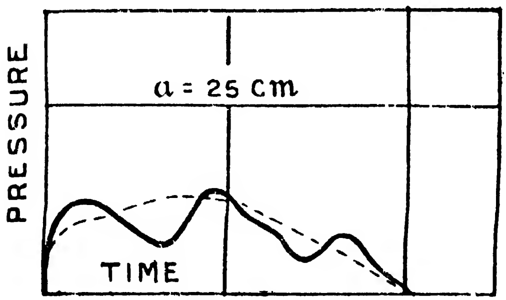


FIG. 19

is very instructive. The continuous curves in Figs. 18 and 19 (taken from paper No. 5) are drawn for an elastic hammer with the help of the exact formulæ of § 1 in this Section. If these are compared with the pressure-time curve for rigid hammer (Fig. 2) we notice at once, that the periodic discontinuities characteristic of the rigid hammer disappear, although the periodic rise of pressure still occurs but in a continuous manner. The sharp angularities give place to well-rounded humps. If the hammer is gradually hardened, so that the value of E grows larger and larger, these humps become more and more prominent until they develop sharp angles again, as is

obvious from Fig. 20, which represents the pressure-time curve during the first epoch only for different values of E/T.

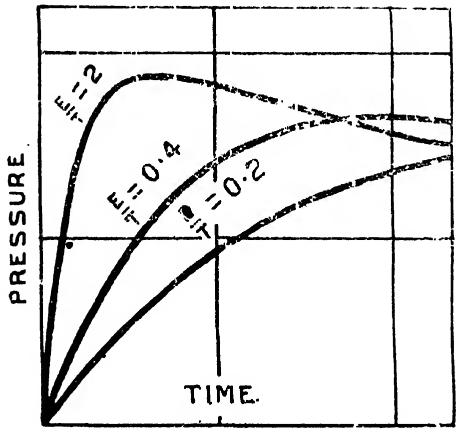


FIG. 20

The dotted curves in Figs. 18 and 19 are drawn with the help of the approximate formula (63) derived by neglecting the vibrations of the striking length a. It is evident that the exact curve can be obtained by superimposing a zig-zag on the approximate curve, just as in the case of the rigid hammer (last paragraph of § 1, Section I). But while the superimposed curve for the hard hammer has sharp angles, that for the elastic hammer is well-rounded in form. Hence in the Fourier series of these curves the terms corresponding to very large order numbers will be of larger amplitude for the rigid hammer than for the elastic hammer. This fact is of great importance, as it helps us to understand the influence of elasticity on the tone of the instrument. The period of the zig-zag for the elastic hammer also corresponds nearly to the fundamental of the striking-length a, so that its form is

$$\sum A_r \sin \frac{\pi r c t}{a'} \qquad \text{.... (65)}$$

where a' is nearly equal to a.

§ 4. *Quality of Tone.*

It is well-known that the quality depends on several factors, the chief amongst which are (1) nature of the material of the hammer, (2) position of the point struck, and (3) striking-velocity of the hammer. We shall investigate here how these factors go to determine the amplitude of the harmonics.

We have seen in § 5 of Section I that a purely analytical expression for the amplitude of the harmonics of a struck string are forbiddingly complicated even for the hard hammer. Since it is much more so for the elastic hammer, we shall content ourselves with the approximate treatment only in studying the quality of tone, taking care not to lose sight however of the general features of the exact theory.

As before, the s^{th} normal co-ordinate ϕ_s is given by

$$\phi_s = \frac{2}{n\rho l} \int_0^t \sin n\,(t-t').\ \mathrm{F} \sin \frac{s\pi a}{l}\,.\ dt',$$

where $n = s\pi c/l$, and F is the value of the pressure. We shall at first take the approximate value of F given by (64). The duration of contact is approximately,

$$t_1 = \frac{1}{\eta c} \tan^{-1} \frac{\mathrm{E}}{\mathrm{T}}.\ \frac{a\eta}{q} \qquad \ldots \quad (66)$$

In order to simplify the integration involved in ϕ_s, we take advantage of the fact that ζct is generally very small and qct_1 is a large negative quantity. So we at once put $e^{\zeta ct} = 1$ and after integration put $e^{\zeta cit_1} = 0$, and get

$$\phi_s = \frac{2}{n\rho l}.\ \frac{\mathrm{V}}{c} \sin \frac{s\pi a}{l}.\ \mathrm{A}_1 \left[-\frac{qc \sin nt + n \cos nt}{q^2 c^2 + n^2} \right]$$

$$+ \frac{2}{n\rho l}.\ \frac{\mathrm{V}}{c} \sin \frac{s\pi a}{l}\,.\ \mathrm{B}_1 \left[\frac{\eta\,(\cos nt_1 - \cos \theta_1)}{n^2 - \eta^2 c^2} \sin nt \right.$$

$$\left. - \frac{\eta \sin nt_1 + n \sin \theta_1}{n^2 - \eta^2 c^2} \cos nt \right] \qquad \ldots \quad (67)$$

$$= a_s \sin nt + b_s \cos nt, \text{ say.}$$

Then the amplitude of the harmonic of order s is given by

$$\sqrt{a_s^2+b_s^2}.$$

It is obvious from the term in (67) having $n^2-\eta^2c^2$ in the denominator that if n or $s\pi c/l$ happens to be very nearly equal to ηc for some value of s, the corresponding s^{th} harmonic will have a very large amplitude. Now in case ρ/m_0 is small, η is approximately given by (62) or

$$\eta=\sqrt{\frac{E}{aE+T}\cdot\frac{\rho}{m_0}-\left(\frac{aE}{aE+T}\right)^2\cdot\frac{\rho^2}{4m_0^2}}.$$

Hence as E/T diminishes from an infinitely large value to a finite one, η also diminishes, so that η is smaller for a soft or elastic hammer than for a hard hammer. Hence the value of s corresponding to the harmonic of the largest amplitude is smaller for a soft hammer than for a hard one. Thus *the primary role of elasticity is to shift the maximum amplitude towards the fundamental.*

We next take into consideration the vibrations of the striking-length a in evaluating ϕ_s. We have seen in article 3 that these vibrations give rise to an additional pressure $\Sigma A_r \sin\frac{\pi rct}{a'}$, so that its contribution to ϕ_s is

$$\frac{2}{n\rho l}\sin\frac{s\pi a}{l}\int_0^t \sin n\,(t-t')\Sigma A_r \sin\frac{\pi rct'}{a'}dt'.$$

If we retain only the most significant terms, it reduces to

$$\frac{1}{n\rho l}\sin\frac{s\pi a}{l}\sin nt\,\Sigma A_r.\frac{1}{n-\frac{\pi rc}{a'}}\left\{\cos\left(n-\frac{\pi rc}{a'}\right)t_1-1\right\}$$

$$-\frac{1}{n\rho l}\sin\frac{s\pi a}{l}\cos nt\,\Sigma A_r.\frac{1}{n-\frac{\pi rc}{a'}}\sin\left(n-\frac{\pi rc}{a'}\right)t_1 \quad \ldots \quad (68)$$

It is obvious from the term $n-\frac{\pi rc}{a'}$ or $\pi c\left(\frac{s}{l}-\frac{r}{a'}\right)$ in the denominator that the harmonics of which the order number s renders

$\pi c\left(\frac{s}{l}-\frac{r}{a'}\right)$ nearly zero, will have considerable intensity. Now a is generally a small fraction of the whole length of the string, so that s is large compared with r. Hence as a rule, only the very high overtones will be reinforced by the vibrations of the striking length a. Thus one of the prime factors in undermining the tone-quality is the vibration of the striking-length.

It has been remarked in section 3 that the co-efficients A_r are generally much larger when the string is struck by a rigid hammer than when it is struck by an elastic one. Since A_r is a factor of (68), it is obvious that the *rigid hammer calls up the higher harmonics to a much greater intensity than the elastic hammer*. This explains the use of the felt-covering on the hammer for the purpose of softening the tone.

In order to study the effect of altering the length a, let us first consider the following special values

$$a=l/s,\ 2l/s,\ 3l/s,\ 4l/s,\ \text{etc.}$$

One sees from the factor $\sin s\pi a/l$ that the expressions (67) and (68) vanish for these values of a. But if a is slightly different from these values the expression (68) is fairly large for those values of r which make $\pi rc/a$ nearly equal to n. Thus we see that as a is varied the overtones pass through a series of maxima separated by small or

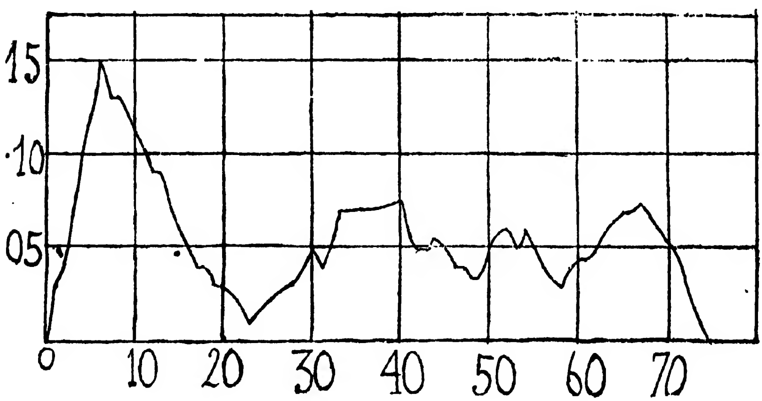


FIG. 21

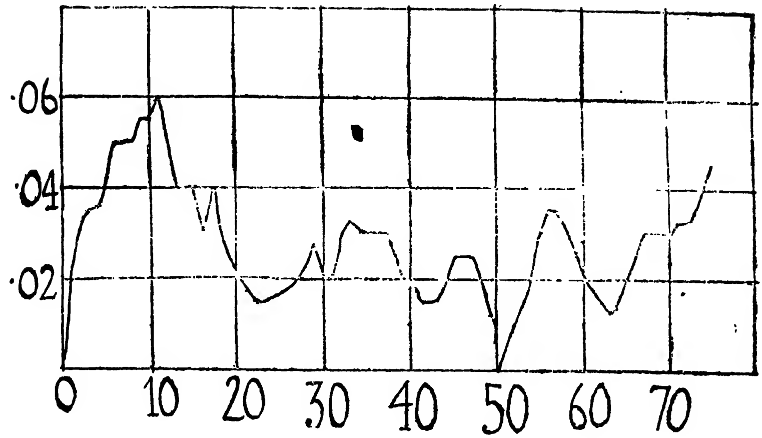


FIG. 22

maximum values. Figs. 21 and 22 taken from Datta's paper illustrate the point.

Lastly, we examine the relation between the initial velocity of the hammer and the intensities of overtones. From the factor V occurring in (67) it is obvious that the *relative* intensities are unaffected by a change in the value of V. But it actually happens that if the hammer undergoes a large compression, the corresponding pressure develops much more rapidly than is to be expected from Hooke's law which is the basis of our theory. This is much the same as if the elasticity co-efficient E does not remain constant, but rather increases with compression. Thus if the hammer has a large velocity and consequently undergoes a large compression, its effective hardness increases. Then from what has been said about the hard hammer one concludes that if the string is struck very hard, the tone loses its softness and acquires a metallic ring.

THE FLORA OF SIKKIM

Gymnospermæ.

BY

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Sikkim is situated between 27° 9′ and 27° 58′ N. Lat. and between 88° 4′ and 88° E. Long., covering an area of about 2,818 sq. miles. The whole of the country is covered by numerous mountain ranges, which rise up to great heights, specially those towards the North-western corner along the frontier of Nepal, and the North-eastern end near Tibet. The elevation gradually decreases in the central region as also in the whole of the frontier along British India. Jannu (25,294), Kabru (24,000), Pandin (22,000), Simvo (23,360), Kanchenjunga (28,146), Siniol-chum (22,620), are the main mountain-peaks in the North-western end, and include among them the three glacial regions, Zemu, Talung and Yalung. In the North-east Kangchenjhau and Chomiomo attain the heights of 22,430 and 22,700 ft., respectively.

Even within an area so small in extent, the climatic condition varies according to elevation, from tropical heat of the lower valley to icy cold of eternal snow, with perfect temperate climate between. The distribution of flora varies likewise and presents within so small a territory, types to be found from the Equator to the Poles. Characteristic landscapes, due to aggregation of particular species over a wide area, though they are few in British India, are however represented in Sikkim by Rhododendron (10,000-14,000 ft.), Bamboo and a few conifers (*Abies Webbiana, Juniperus, Tsuga Brunoniana* and *Picea*), which may constitute either pure formations or are associated with others—the conifers or rhododendron forming a mixed vegetation.

Of the four great divisions of the Gymnosperms Ginkgo alone is absent from the Indian flora while others are more or less represented, although very few of these can be claimed as endemic, mostly

being either of Tibetan or of Chinese element. Coniferae as a whole, which constitute the characteristic vegetation of the temperate regions of both hemispheres, are confined in India mainly along the temperate and the alpine Himalayas extending as far in the east as the offshoots of the great mountain ranges in Assam and Burma. The Deccan, however, is conspicuous by the absence of conifers, showing a close bearing on the glacial theory of the formation of the Deccan and the isolated nature of its flora.

India contains about 13 genera and 25 species of conifers, of which Sikkim alone is represented by 9 genera and 14 species, most of the genera being monotypic and confined between 9,000-12,000 ft.; but *Tsuga Brunoniana*, *Juniperus recurva* and *Pseudo salina* ascend high in the alpine zone beyond the limit of trees. *Abies Webbiana*, *Picea Morinda* and *Larix Griffithii* are more gregarious, while *Pinus Khasya*, *Juniperus salina*, *Cupressus funebris* and *C. torulosa* are usually planted.

Cycads contain 9 genera and 75 species, exclusively confined to the tropical and sub-tropical regions; India, however, is represented by a solitary genus and a few species. *Cycas pectina* is distributed in various parts of India as also in Sikkim, in the *sál* forest of Sikkim Terai and in the great Rangit Valley. *Gnetum scandens* is the sole representative of the group while 2 species with a variety of Ephedra have been reported from Sikkim.

CONIFERAE.

Trees or shrubs, usually with resin, branches often whorled. Leaves rigid, linear, subulate or scale-like, often in tuft. Male flowers in deciduous catkin bearing numerous antheriferous scales with two or more (Cupressus, etc.) anther cells on the under surface. Female flowers in cones consisting of ovuliferous scales arranged spirally or decussately on the axis. Endosperm oily, cotyledons two or more.

A key to the Conifers of Sikkim.

(A) Leaves, at least those of the terminal branchlets, densely crowded and overlapping.

(X) Leaves all of one shape.

(x) Leaves subulate, adpressed.

A small tree or an erect or prostrate shrub, colour glaucous; many branches parallel to the stem, decurved and ascending with pendulous branchlets.

Juniperus recurva, Hamilton.

N. W. Himalaya, Himalaya, Kashmir to Bhutan, altitude 7,500-15,000 ft.

(y) Leaves ovate or ovate-oblong adpressedly imbricate and decurrent. Tall trees.

(*) Leaves with glands.

(@) Crown broad-pyramidal; branches spreading, tips pendulous, leaves triangular-ovate, obtuse.

Cupressus torula, Don.

Outer range of the Himalaya from Chamba to Nepal, alt. 6,000-9,000 ft.

(@@) Crown narrow cylindric; branches and their tips erect; leaves ovate-oblong. Trees planted.

Cupressus sempervirens, Lin.

N. W. Himalaya, Afganistan.

(**) Leaves without gland.

Crown broadly pyramidal, branches horizontal, branchlets pendulous, distichous, compressed. Trees planted.

Cupressus funebris, Endlicher.

Indigenous in Ichang (China) planted in Nepal, Sikkim and Bhutan.

(Y) Leaves heterophyllous, those of upper branches and of branchlets differing from those of lower branches.

(*) Leaves without a gland, of lower branches linear and pungent, of terminal branchlets tetrafarious, adpressed imbricate. A bush or a tree.

Juniperus pseudosalina, Fischer and Meyer.
Himalaya, Nepal and Bhutan, alt. 9,000-15,000 ft.

(**) Leaves with a large dorsal gland, of lower branches subulate and pungent, of upper branches and branchlets scale-like.

Juniperus macropoda, Boissier.
Baluchistan, Kuram valley, Chitral, N. W. Himalaya and Nepal, alt. 5,000-14,000 ft.

(B) Leaves bifarious or scattered or if close then not overlapping.

(X) Leaves scattered or distichous, not in cluster.

(x) Leaves scattered.

(*) Leaves 20-30 c.m. long, narrowed at base, midrib prominent on both surfaces, linear-lanceolate, petiole 4-13 m.m. long. An ever-green tree.

Podocarpus neriifolia, Don.
Nepal, Assam, Chittagong, Tenasserim, etc., alt. 3,000 ft.

(**) Leaves 4 c.m. long and less pungent, conical crown.

(@) Leaves four-sided jointed at base at least 2 c.m. long.

Picea Morinda, Link.
Himalaya from Kashmir to Garhwal, Bhutan and Nepal, alt. 8,000-15,000 ft.

(@@) Leaves laterally compressed, decurrent, less than 1·5 c.m. long, incurved. Trees planted.

Cryptomeria japonica, Don.
China, Japan.

(Y) Leaves bifarious or distichous, pale-green or white.

(*) Leaves linear pale-green beneath, branches spreading.

Taxus bacatta, Linn.

Himalaya, Nepal, Bhutan, Khasia Hills, Upper Burma, alt. 6,000-10,000 ft.

(**) Leaves white beneath.

(@) Margin of leaves recurved, tips serrulate. Tall trees.

Tsuga Brunoniana, Carr.

N. W. *Himalaya,* Nepal and *Bhutan.* alt. 6,000-10,000 ft.

(@@) Leaves flat, tips usually retuse or bicuspidate, rarely round or sub-acute. Tall dark coloured trees.

Abies Webbiana, Lind.

N. W. and E. Himalaya, alt. 8,000-14,000 ft.

(Y) Leaves at least the secondary ones in clusters.

(*a*) Cluster consisting of 2-8 leaves surrounded at their base by a sheath of hyaline scales and standing at the axile of primary membranous scales.

(@) Leaves in cluster of 5-8, 10-20 c.m. long, filiform, greyish or bluish green with serrate margin, both surfaces white; sheaths and primary leaves deciduous. A lofty tree.

Pinus excellsa, Wallich.

Himalaya, Nepal, Bhutan, Afganistan, alt. 6,000-12,500.

((@@) Leaves in cluster of three, 20-30 c.m. long, slender dark or light green, serrulate, sheaths persistent. A large tree.

Pinus longifolia, Roxburgh.

Sub and outer Himalaya, alt. 1,500-6,000 ft.

(*b*) Cluster consisting of 30-50 flat linear slender leaves; about 25 m.m. long, pale-green. A small or a middle-sized tree.

Larix Griffithii, Hooker.

E. Nepal, Bhutan, alt. 8,000-12,000 ft.

CYCADACEAE.

Small trees, stem cylindric usually short, sometimes branched. primary root usually a long tap root. Leaves large pinnate, palm like, alternate with whorls of short coriaceous scales forming terminal crowns. Flower dioecious usually in form of cones, consisting of central axis bearing numerous thickened flat or variously formed peltate scales or sporophylls; the lowest whorl usually sterile. In male cone scales usually bear 2-6 globose sporangia (pollen-sac) on their under-surfaces: often arranged in stellate groups; dehiscence by longitudinal slits. Carpellary leaves in cones (except in Cycas) crowded round the apex of the stem; bearing usually two sporangia. Ovules large orthotropus with one integument. Seeds large, full of endosperm, nucellus, much reduced: testa two-layered, outer fleshy and inner woody. Cotyledons two, usually united at tips.

Stem 4-8 ft., leaf glabrous 4-5 ft. long carpophylls densely tawny with strong subulate spiny teeth, ovules 4-6 glabrous. Seeds ovoid, glabrous, orange-red or yellow.

Cycas pectinata, Griff.

Burma, Nepal, Khasia Hill, Manipur, Chittagong.

GNETACEAE.

Shrubs, small trees or climbers. Stem and branches without resin, jointed at nodes. Leaves opposite, decussate or whorled, exstipulate, large, or reduced to 3-4 dentate scaly sheath. Flowers unisexual, male flower with 2-8 anthers enclosed by bracts. Female flower has an erect ovule with one integument enclosed by a single or double perianth or bracts which become fleshy in fruits.

(x) Leaves large ever-green, opposite, entire.
A large ever-green climber, leaves elliptic, venation reticulate. Fruit covered with silvery scales when young, with short stalk.

Gnetum scandens, Roxb.

Assam, Burma, Chittagong, hills of Deccan.

(Y) Leaves membranous, forming sheaths at the base of internodes.

(@) A low shrub, branches short in tuft, male spike usually solitary, flowers in 2-4 pairs, anthers 5-8, female flowers usually solitary, bracts 2-3 pairs. Fruit red when ripe, edible.

Ephedra Vulgaris, Rich.

(@@) An erect shrub, branches rather stout often striate, bracts connate, anther 6-8, sessile, crowded, large.

Ephedra Pachyclada, Boiss.

CALCUTTA, May, 1929.

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On the free vibrations of a gas enclosed in a rigid cylinder of elliptic section.

BY

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INTRODUCTION.

The Vibrations[1] of a gas in a rigid cylinder of circular section had long been considered. The author of the present paper is not aware of any other paper dealing with the vibrations of a gas in a rigid cylinder of elliptic section. The problem is analogous to the transverse vibrations of an elliptic membrane, a complete solution of which was given by M. Mathieu ;[2] the principal mathematical difference between the two questions lies in the fact that the boundary-conditions of the two problems are expressed by two different equations. After the success of E. L. Ince[3] in having the elliptic cylinder functions of the second kind, corresponding to the Mathieu-functions, it is not difficult to consider the vibrations of a gas within two rigid confocal elliptic cylinders, which I have discussed in Section VI. Finally, in Section VII, I have considered the transverse vibrations of a membrane bounded by two confocal ellipses.

I

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, be the cross-section of the elliptic cylinder which we suppose to be infinitely long so that the motion will be a two-dimensional one, as being transverse. The gas being assumed to be

[1] The symmetrical vibrations within a cylindrical boundary were considered by Duhamel (Liouville, Jour. Math., Vol. 14, p. 66, 1849).

See Theory of Sound, Vol. II, p. 297—Lord Rayleigh. Also, the Dynamical Theory of Sound, p. 263 (2nd ed.)—H. Lamb.

[2] Liouville, XIII, 1868 ; Cours de physique Mathematique, 1873, p. 122.

[3] E. L. Ince, Proc. Edin. Math. Soc., Vol. 33, p. 2.

frictionless, if ϕ denote the velocity-potential of the motion within the contemplated space, it must satisfy the wave-equation

$$(i) \quad \frac{\partial^2\phi}{\partial t^2}=c^2\nabla_1{}^2\phi,$$

where c is the wave-velocity and

$$\nabla_1{}^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},$$

(*ii*) The boundary condition $\frac{\partial\phi}{\partial n}=0$, dn=element of normal and (*iii*) its first and second differential co-efficients must be finite and continuous everywhere within the space.

II

If $\frac{2\pi}{kc}$ denote the period of vibration and ϕ varies as e^{ikct}, the partial differential equation $\frac{\partial^2\phi}{\partial t^2}=c^2\nabla_1{}^2\phi$, would be reduced to

$$(\nabla_1{}^2+k^2)\phi=0$$

Let $x+iy=h\cosh(\xi+i\eta)$

so that (x, y) denotes the Cartesian co-ordinates of a point of which the elliptic co-ordinates are (ξ, η), the origin being the centre of the elliptic section, the axes of co-ordinates, the axes of the ellipse and h, the distance of a focus from the center.

We then have

$$x=h\cosh\xi\cos\eta$$

$$y=h\sinh\xi\sin\eta$$

and $h=ae=a'e'$

where

e = eccentricity of the elliptic section

and a, e' = respectively the semi-major axis and eccentricity of the confocal ellipse passing through the point under consideration,

so that

ξ = Const. are confocal ellipses

and η = const. are confocal hyperbolas.

Let us change the independent variables by the scheme given above, then the two-dimensional equation $(\Delta_1{}^2+k^2)\phi=0$, would be reduced to

$$\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}+h^2k^2\,(\cosh^2\xi-\cos^2\eta)=0.$$

To obtain elementary solutions of the above partial differential equation in product-forms, let us assume

$$\phi=F(\xi)\;G(n),$$

where $F(\xi)$ = a function of ξ only

and $G(\eta)$ = a function of η only.

Substituting in the above differential equation

we get

$$\frac{\partial^2 F}{\partial\xi^2}+(h^2k^2\cosh^2\xi-A)F=0$$

$$\text{and}\quad\frac{\partial^2 G}{\partial\eta^2}-(h^2k^2\cos^2\eta-A)G=0$$

where A is a constant.

The above equations can be written in the forms,

$$\frac{\partial^2 F}{\partial\xi^2}-\left(A-\frac{h^2k^2}{2}-\frac{h^2k^2}{2}\cosh 2\xi\right)F=0$$

$$\text{and}\quad\frac{\partial^2 G}{\partial\eta^2}+\left(A-\frac{h^2k^2}{2}-\frac{h^2k^2}{2}\cos 2\eta\right)G=0.$$

Put

$\xi = -iz$, in the first equation

$\eta = z$, in the second equation

$$a = A - \frac{h^2k^2}{2},$$

$$\text{and } q = -\frac{h^2k^2}{32}.$$

Then, both of them would reduce to Mathieu's equation

$$\frac{\partial^2 U}{\partial z^2} + (a + 16\, q \cos 2z)U = 0.$$

III

In most of the physical (as distinguished from astronomical) problems, only periodic solutions, or what are called Mathieu-functions (even or odd) of Mathieu's equation are wanted. These solutions are infinite in number and are written, introducing Professor Whittaker's notation, using capital letters as

$Ce(z)$, $Ce_1(z)$, $Ce_2(z)$,.....................$Ce_n(z)$..........................

$Se_1(z)$, $Se_2(z)$.......................$Se_n(z)$..........................

we observe that the above differential equation, *viz.*, Mathieu's equation, is a linear differential equation with a periodic co-efficient which is a single-valued function of the independent variable. Floquet[1] has given an analytical investigation of the nature of the general solution of equations of this type, which, previous to the publication of his theory, was otherwise perceived by astronomers from circumstantial inferences. According to Floquet's theory, the general solution of Mathieu's equation must be of the form

$$U = Ae^{\mu z}\,\psi(z) + Be^{-\mu z}\,\psi(-z)$$

where

$\psi(z)$ and $\psi(-z)$ are periodic functions having the same period as the co-efficient of the differential equation, *i.e.*, of Mathieu's equation,

[1] Ann. de l'Ecole norm. Sup. (2) XII (1883), p. 43.

Vide Whittaker and Watson—Modern Analysis (3rd ed.), p. 412.

μ is a constant, being a definite function of the constants of the original differential equation, and, A and B are the arbitrary constants of the solution.

When the constants of Mathieu's equation are such that $\mu=0$, the above solution fails to give the general solution, the function $\psi(-z)$ ceasing to be distinct from the function $\psi(z)$; in this case, one solution is purely periodic, *viz.*, the Mathieu-functions, and to obtain the corresponding second solution called 'elliptic cylinder functions of the second kind' which however are not periodic, two distinct methods have been advanced by E. L. Ince[1] who introduces the notations, as suggested by Professor Whittaker

$$in_0(z),\ in_1(z),\ in_2(z),\ldots\ldots\ldots\ldots in_n(z)\ldots\ldots\ldots$$

$$Jn_1(z),\ \ Jn_2(z),\ldots\ldots\ldots\ldots Jn_n(z)\ldots\ldots\ldots$$

for representing them.

Professer Whittakar[2] has given a very powerful and elegant method, *viz.*, the "method of change of parameters," as he calls it, for obtaining a general solution, in Floquet's form, of Mathieu's equation, reducing as special cases to the Mathieu-functions $Ce_1(z)$, $Se_1(z)$; the method can very easily be extended to provide expansions reducing to any required Mathieu-function as a special case. The efficacy of the method is best understood when numerical calculations are involved, subject to the condition that $|\ q\ |$ is small. It would be proper here to make, by way of illustration, a statement of the solution, reducing to $Ce_1(z)$, $Se_1(z)$, as degenerate cases, which Professor Whittaker obtained as a result of exploring the possibility of a solution of Mathieu's equation in the form

$$U=e^{\mu z}\ \phi(z).$$

$\phi(z)$ has been obtained in the form

$$\phi(z)=\sin(z-\sigma)+a_3\ \cos(3z-\sigma)+b_3\ \sin(3z-\sigma)+a_5\ \cos(5z-\sigma)$$
$$+b_5\ \sin(5z-\sigma)+\ldots\ldots\ldots\ldots\ldots\ldots,$$

the parameter σ being rendered definite by the fact that no term in

1 E. L. Ince. Proc. Edin. Math. Soc., Vol. 33, p. 2.
2 Proc. Edin. Math. Soc., Vol. 32 (1913-14), p. 75.

cos $(z-\sigma)$ is to appear in $\phi(z)$, with the following approximations valid for small values of q and real values of σ :—

$$\mu=4q \sin 2\sigma-12q^3 \sin 2\sigma-12q^4 \sin 4\sigma \text{ ....}$$

$$a=1+8q \cos 2\sigma+(-16+8 \cos 4\sigma)q^2-8q^3 \cos 2\sigma$$
$$+(\tfrac{256}{3}-88 \cos 4\sigma)q^4+\dots \quad \dots\dots$$

$$a_3=3q^2 \sin 2\sigma+3q^3 \sin 4\sigma+(-\tfrac{274}{9} \sin 2\sigma+9 \sin 4\sigma)q^4+\dots\dots\dots$$

$$b_3=q+q^2 \cos 2\sigma+(-\tfrac{14}{3}+5\cos 4\sigma)q^3+(-\tfrac{74}{9} \cos 2\sigma$$
$$+7\cos 6\sigma)q^4+\dots$$

$$a_5=\tfrac{14}{9}q^3 \sin 2\sigma+\tfrac{44}{27}q^4 \sin 4\sigma+\dots\dots\dots\dots\dots\dots$$

$$b_5=\tfrac{1}{3}q^2+\tfrac{4}{9}q^3 \cos 2\sigma+(-\tfrac{155}{54}+\tfrac{82}{27} \cos 4\sigma)q^4+\dots\dots\dots\dots,$$

The general solution for the case under consideration is obtained by changing the sign of σ in the above value for u and then adding up the two. Thus, we have,

$$U=A\Lambda(z, 0, q)+B\Lambda(z, -\sigma, q),$$

in perfect agreement with Floquet's theory.

IV

It appears from section III that the solution of Mathieu's equation (which is only a particular case of Hill's equation[1]) would be given by

$$U=Ce_0(z, q),\ Ce_1(z, q),\ Ce_2(z, q)\dots\dots\dots Ce_n(z, q)\dots\dots\dots$$
$$Se_1(z, q),\ Se_2(z\ q)\dots\dots\dots Se_n(z, q)\dots\dots\dots,^2$$

if only periodic solutions are sought, and generally

$$U=A_n\ Ce_n(z, q)+B_m\ in_n(z, q)$$
$$Se_n(z, g) \quad Jn_n(z, q),$$

where n may have all values from 0 to infinity. Consequently, since $G_1(\eta)$ must be a purely periodic solution,

[1] Acta. Math. VIII (1886).

$$G(\eta)=Ce_0(\eta,q),\ Ce_1(\eta,q),\ Ce_2(\eta,q)\ldots\ldots Ce_n(\eta,q)\ldots\ldots$$

$$Se_1(\eta,q),\ Se_2(\eta,q)\ldots\ldots Se_n(\eta,q)\ldots\ldots$$

and

$$F(\xi)=A_n\ Ce_n(i\xi,q)+B_n\ In_n(i\xi,q)$$

$$Se_n(i\xi,q)\quad Jnn(i\xi,q),$$

where n may have all values from 0 to infinity. Finally, we have, as particular solutions,

$$\phi=Ce_n(n,q)[ACe_n(i\xi,q)+B\ In_n(i\xi,q)]e^{ikct},$$

$$Se_n(n,q)[ASe_n(i\xi,q)+B\ Jn_n(i\xi,q)]e^{ikct},$$

where n may have all values from zero to infinity.

V

To apply the above solution to the problem of the vibrations of a gas contained in an infinite rigid cylinder of elliptic section, we see that ϕ, $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial y}$ vary in a continuous manner throughout the whole area of the elliptic section. Consider two points m and m' symmetrical with respect to the right line joining the foci and very close to one another, ξ of the two points would be the same and very small but η of the two points would be equal and of opposite signs. We have,

$$\frac{\partial V}{\partial\eta}=-\frac{\partial V}{\partial x}h\cosh\xi\sin\eta+\frac{\partial V}{\partial y}h\sinh\xi\cos\eta.$$

$$\frac{\partial V}{\partial\xi}=\frac{\partial V}{\partial x}h\sinh\xi\cos\eta+\frac{\partial V}{\partial y}h\cosh\xi\sin\eta.$$

Therefore, if ξ is zero or infinitely small, we may write,

$$\frac{\partial V}{\partial x}=-\frac{1}{h\cosh\xi\sin\eta}\frac{\partial V}{\partial\eta},$$

and, $$\frac{\partial V}{\partial y}=\frac{1}{h\cosh\xi\sin\eta}\frac{\partial V}{\partial\xi}.$$

ϕ, $\dfrac{\partial \phi}{\partial x}$ and $\dfrac{\partial \phi}{\partial y}$ of the two points m and m' must differ by an infinitely small amount from one another and be equal to one another when their ξ vanishes. Therefore, representing generally ϕ by $\phi(\xi, \eta)$, we have,

$$\phi(o, \eta) = \phi(o, -\eta),$$

$$\frac{\partial}{\partial \eta}\phi(o, \eta) = \frac{\partial}{\partial \eta}\phi(o, -\eta); \quad \frac{\partial}{\partial \xi}\phi(o, \eta) = -\frac{\partial}{\partial \xi}\phi(o, -\eta) \ldots\ldots \quad (a).$$

If, now, we choose any one of the particular solutions given in Section (IV), the equations (a) give, when the solution $Ce_n(\eta, q)$ $[ACe_n(i\xi, q) + Bin_n(i\xi, q)]$ is taken,

$$ACe_n(\eta, q)Ce_n(o, q) = ACe_n(\eta, q)Ce_n(o, q),$$

and $$BCe_n(\eta, q)in'_n(o, q) = -BCe_n(\eta, q)in'_n(o, q)$$

since $$in_n(i\xi, q) = 0, \qquad \text{for } \xi = 0$$

and $$Ce'_n(i\xi, q) = 0, \qquad \text{for } \xi = 0,$$

whence it follows that the constant B is zero.

Again, if we choose the other solution, $Se_n(\eta, q)[Se_n(i\xi, q)c + DJn_n(i\xi, q)]$, it can be similarly proved with the help of the equations (a) that the constant D is zero. Thus the appropriate solutions[1] for the particular problem we have in view are

$$\phi = ACe_n(\eta, q)\, Ce_n(i\xi, q)e^{ikct},$$

$$CSe_n(\eta, q)\, Se_n(i\xi, q)\, e^{ikct},$$

where A and C are two constants and n may have all values from zero to infinity.

[1] M. Mathieu obtained similar results by following the same line of argument —Cours de physique Mathematique, pp. 139, 76, 77, and we have adopted the above from Mathieu.

If dn denote an element of normal of the ellipse $\xi=$const. at any point whose Cartesian co-ordinates are (x, y) given by the scheme in Section (II), we have

$$dn=\frac{d\xi}{h}, \qquad \text{where}$$

$$\frac{1}{h^2}=\left(\frac{\partial x}{\partial \xi}\right)^2+\left(\frac{\partial y}{\partial \xi}\right)^2.$$

Therefore the boundary condition $\frac{\partial \phi}{\partial n}=0$ is reduced to

$$\frac{\partial \phi}{\partial \xi}=0.$$

Let us suppose that q is so small that in the Mathieu-functions which determine the value of the velocity-potential of the motion within the cylinder, we need retain only the first power of q. On this supposition, taking

$$\phi=\mathrm{A}Ce_k(\eta, q)\ Ce_n(i\xi, q)e^{ikct}$$

$$\mathrm{C}\ Se_n(\eta, q)\ Se_n\ (i\xi, q)^{ikct}$$

and $\xi=\xi_0$, as the elliptic section of the cylinder within which motion takes place, so that $\cosh \xi_0=\frac{1}{e}$, e denoting the eccentricity of the ellipse, and using the reduced boundary condition written above, we have the equations

$$C'e_n(i\xi_0, q)=0$$

$$S'e_n(i\xi_0, q)=0,$$

to determine the corresponding frequency of vibration.

Let us choose, in particular, the functions $C'e_1\ (i\xi_0, q)$ and $Se_1(i\xi_0, q)$. Then from above, we have (i)

$$\sinh\xi_0+3q\sinh\xi_0=0,$$

whence

$$q = -\frac{h^2k^2}{32} = -\frac{\sinh\xi_0}{3.\ \sinh 3\xi_0} = -\frac{1}{3}.\frac{1}{4\cosh^2\xi_0 - 1}$$

$$= -\frac{1}{3}.\frac{e^2}{4-e^2}$$

$$\therefore\quad kh = 4\sqrt{\frac{2}{3}}.\frac{e}{\sqrt{4-e^2}}.$$

and the corresponding frequency is given by

$$f = \frac{kc}{2\pi} = kh.\frac{c}{2\pi h} = \frac{c}{2\pi h}.4\sqrt{\frac{2}{3}}.\frac{e}{\sqrt{4-e^2}} = \frac{2ce}{\pi h}\sqrt{\frac{2}{3(4-e^2)}}$$

and (*ii*) $\qquad \cosh\xi_0 + 3q\cosh 3\xi_0 = 0$

$$\therefore\quad q = -\frac{h^2k^2}{32} = -\frac{1}{3}.\frac{\cosh\xi_0}{\cosh 3\xi_0} = -\frac{1}{3}\frac{1}{4\cosh^2\xi_0 - 3}$$

$$= -\frac{1}{3}.\frac{e^2}{4-3e^2}.$$

$$\therefore\quad kh = 4\sqrt{\frac{2}{3}}.\frac{e}{\sqrt{4-3e^2}}.$$

and the corresponding frequency is accordingly

$$f = \frac{kc}{2\pi} = \frac{2ec}{\pi h}.\sqrt{\frac{2}{3(4-3e^2)}}$$

If we retain the second power of q also, the frequency-equation $Ce'_1(i\xi_0, q) = 0$, gives

$$\sinh\xi_0 + 3q\sinh 3\xi_0 + q^2[\tfrac{5}{3}\sinh 5\xi_0 - 3\sinh 3\xi_0] = 0.$$

since, $\qquad \sinh 3\xi_0 = \sinh\xi_0\left(\frac{4}{e^2} - 1\right),$

and $\frac{5}{3}\sinh 5\xi_0 - 3\sinh 3\xi_0 = \sinh\xi_0 \left[\frac{14}{3} - \frac{32}{e^2} + \frac{80}{3}\cdot\frac{1}{e^4}\right]$,

the above equation reduces to

$$q^2(14e^4 - 96e^2 + 80) + 9qe^2(4 - e^2) + 3e^4 = 0,$$

whence by successive approximation

$$q = -\frac{h^2k^2}{32} = -\frac{e^2}{81}\cdot\frac{41e^4 - 312e^2 + 512}{(4 - e^2)^3}$$

Therefore

$$hk = \frac{4e}{9}\sqrt{\frac{2(41e^4 - 312e^2 + 512)}{(4 - e^2)^3}}$$

VI

In the case of the vibrations of a gas enclosed between two infinite confocal cylinders, the appropriate particular solutions of the two-dimensional wave-equation $\ddot{\phi} = c^2 \triangle_1{}^2 \phi$, are

$$\phi = \mathrm{Ce}_n(\eta, q)[\mathrm{A}\,\mathrm{Ce}_n(i\xi, q) + \mathrm{B}\,in_n(i\xi, q)]e^{ikct},$$

$$\mathrm{Se}_n(\eta, q)[\mathrm{A}\,\mathrm{Se}_n(i\xi, q) + \mathrm{B}\,\mathrm{J}n_n(i\xi, q)]e^{ikct},$$

where n may have all values from o to infinity and $in_n(i\xi, q)$, $\mathrm{J}n_n(i\xi, q)$ are elliptic cylinder functions of the second kind, as obtained by E. L. Ince, corresponding to the Mathieu-functions.

Let the two confocal ellipses be given by $\xi = \xi_0$ and $\xi = \xi_1$, which form respectively the internal and external boundaries.

The conditions to be satisfied on the two boundaries are

$$\frac{\delta\phi}{\delta\xi} = 0, \qquad \text{on} \quad \xi = \xi_0$$

and $$\frac{\delta\phi}{\delta\xi} = 0, \qquad \text{on} \quad \xi = \xi_1.$$

From the above boundary conditions, we have

$$\left.\begin{aligned} &\text{A}C'e_n(i\xi_0, q)+\text{B}in'_n(i\xi_0, q)=0 \\ \text{and} \quad &\text{A}C'en(i\xi_1, q)+\text{B}i'n_n(i\xi_1, q)=0 \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{or} \quad &\text{A}S'en(i\xi_0, q)+\text{B}Jn'_n(i\xi_0, q)=0 \\ \text{and} \quad &\text{A}S'en(i\xi_1, q)+\text{B}Jn'_n(i\xi_1, q)=0 \end{aligned}\right\},$$

according as we choose the solution

$$\phi=\text{C}e_n(\eta, q)[\text{A}\text{C}e_n(i\xi, q)+\text{B}\, in_n(i\xi, q)]e^{ikct},$$

$$\text{or} \quad \phi=\text{S}e_n(\eta, q)[\text{A}\text{S}e_n(i\xi, q)+\text{B}Jn_n(i\xi, q)]e^{ikct},$$

to determine the ratio A : B, and q and therefore the frequency. Eliminating A and B, we have

$$\begin{vmatrix} C'e_n(i\xi_0, q) & in'_n(i\xi_0, q) \\ C'e_n(i\xi_1, q) & i'n_n(i\xi_1, q) \end{vmatrix}=0,$$

or

$$\begin{vmatrix} S'e_n(i\xi_0, q) & J'n_n(i\xi_0, q) \\ S'e_n(i\xi_1, q) & J'n_n(i\xi_1, k) \end{vmatrix}=0$$

Let us retain only the first power of q in $C'e_n(i\xi_0, q)$, $C'e_n(i\xi_1, q)$, $S'e_n(i\xi_0, q)$, $S'e_n(i\xi_1, q)$, $i'n_n(i\xi_0, q)$, $i'n_n(i\xi_1, q)$, $J'n_n(i\xi_0, q)$, and $J'n_n(i\xi_1, q)$. Choosing in particular, $n=1$, we have,

$$C'e_1(i\xi_0, q)=-i[\sinh \xi_0+3q \sinh 3\xi_0],$$

$$C'e_1(i\xi_1, q)=-i[\sinh \xi_1+3q \sinh 3\xi_1],$$

$$S'e_1(i\xi_0, q)=\cosh \xi_0+3q \cosh 3\xi_0,$$

$$S'e_1(i\xi_1, q)=\cosh \xi_1+3q \cosh 3\xi_1,$$

$$i'n_1(i\xi_0, q)=-8q\,(\xi_0 \sinh \xi_0+\cosh \xi_0)+\cosh \xi_0+3q \cosh 3\xi_0,$$

$$i'n_1(i\xi_1, q)=-8q(\xi_1 \sinh \xi_1+\cosh \xi_1)+\cosh \xi_1+3q \cosh 3\xi_1,$$

$$J'n_1(i\xi_0, q) = -i[8q(\xi_0 \cosh \xi_0 + \sinh \xi_0) + \sinh \xi_0 + 3q \sinh 3\xi_0],$$

$$J'n_1(i\xi_1, q) = -i[8q(\xi_1 \cosh \xi_1 + \sinh \xi_1) + \sinh \xi_1 + 3q \sinh 3\xi_1].$$

Therefore, from the first of the above determinants we have,

$$0 = (\sinh \xi_0 + 3q \sinh 3\xi_0)[\cosh \xi_1 + 3q \cosh \xi_1 - 8q(\xi_1 \sinh \xi_1 + \cosh \xi_1)]$$
$$-(\sinh \xi_1 + 3q \sinh 3\xi_1)[\cosh \xi_0 + 3q \cosh \xi_0 - 8q(\xi_0 \sinh \xi_0 + \cosh \xi_0)]$$

Whence rejecting the square of q, and simplifying, we get

$$q = -\tfrac{1}{4}\frac{1}{3\cosh \xi_0 \cosh \xi_1 \cosh(\xi_1 + \xi_0) + 2(\xi_1 - \xi_0)\dfrac{\sinh \xi_1 \sinh \xi_0}{\sinh(\xi_1 - \xi_0)} - 2}$$

Therefore,

$$kh = \frac{2\sqrt{2}}{\sqrt{3\cosh \xi_0 \cosh \xi_1 \cosh(\xi_0 + \xi_1) + \dfrac{2\sinh \xi_0 \sinh \xi_1 (\xi_1 - \xi_0)}{\sinh(\xi_1 - \xi_0)} - 2}}$$

The corresponding frequency is given by

$$f = \frac{kc}{2\pi} = \frac{c}{2\pi h}(kh),$$

where kh is given above.

(a) When $\xi_1 - \xi_0$ is negligibly small, so that we have a cylindrical sheet of elliptic section, the corresponding frequency is given by

$$f = \frac{c}{2\pi h}(kh) = \frac{c}{2\pi h} \cdot \frac{2\sqrt{2}.e^2}{\sqrt{6 - e^2 - 4e^4}},$$

where e is the eccentricity of the internal boundary.

(b) An interesting case occurs when we choose the internal boundary to be $\xi_0 = 0$. This corresponds to the case of vibration of a gas enclosed in an elliptic cylinder with a rigid plane partition stretching from one focus to the other. The effect of this partition is to render possible a difference of pressure on its two sides. If there be no difference of pressure, the partition can be removed and the vibrations

would be the same as in a complete elliptic cylinder. If, however, a difference of pressure exists, the corresponding frequency for the particular case we have considered above is given by

$$f=\frac{kc}{2\pi}=\frac{c}{\pi h}\cdot\sqrt{\frac{2}{3-2e^2}},$$

where e is the eccentricity of the external boundary.

VII

The solution of the problem of vibration of a membrane bounded by two confocal ellipses is no longer difficult. If z denote the displacement of a point, normal to the undisturbed position of the membrane, then z, as is well-known, must be a solution of

$$\frac{\partial^2 z}{\partial t^2}=c^2\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right),$$

satisfying the condition $z=0$, on either of the boundaries $\xi=\xi_0$, $\xi=\xi_1$.

We would, accordingly, have as particular solutions,

$$z=ce_n(\eta,\ q)[\mathrm{A}ce_n(i\xi,\ q)+\mathrm{B}in_n(i\xi,\ q)]e^{ikct},$$

$$\mathrm{S}e_n(\eta,\ q)[\mathrm{A}\ se_n(i\xi,\ q)+\mathrm{BJ}n_n(i\xi,\ q)]e^{ikct},$$

of which the ratio A : B and the value of q (and therefore of the frequency) are to be found from the two boundary conditions stated above. If, as in the previous section, we choose the solution,

$$z=ce_1(\eta_1 q)[\mathrm{A}ce_1(i\xi_1 q)+\mathrm{B}in_n(in\xi,q)]e^{ikct},$$

in particular, we get, from the two boundary conditions, neglecting, as before, powers of q beyond the first,

$$0=\mathrm{A}[\cosh\xi_0+q\cosh 3\xi_0]+\mathrm{B}i\,[\sinh\xi_0+q\sinh 3\xi_0-8q\xi_0\cosh\xi_0]$$

and

$$0=\mathrm{A}[\cosh\xi_1+q\cosh 3\xi_1]+\mathrm{B}i\,[\sinh\xi_1+q\sinh 3\xi_1-8q\xi_1\cosh\xi_1]$$

Eliminating A and B, we have,

$$O=\begin{vmatrix}\cosh \xi_0+q\cosh 3\xi_0 & \sinh \xi_0+q\sinh 3\xi_0-8q\xi_0\cosh \xi_0\\ \cosh \xi_1+q\cosh 3\xi_1 & \sinh \xi_1+q\sinh 3\xi_1-8q\xi_1\cosh \xi_1\end{vmatrix}$$

whence simplifying and rejecting squares of q we get

$$q=-\frac{1}{4}\,\frac{1}{\cosh(\xi_1+\xi_0)\cosh(\xi_1-\xi_0)-\dfrac{2(\xi_1-\xi_0)\cosh\xi_1\cosh\xi_0}{\sinh(\xi_1-\xi_0)}}$$

$$=-\frac{h^2k^2}{32}$$

$$\therefore\quad hk=\frac{2\sqrt{2}}{\sqrt{\cosh(\xi_1+\xi_0)\cosh(\xi_1-\xi_0)-\dfrac{2(\xi_1-\xi_0)\cosh\xi_1\cosh\xi_0}{\sinh(\xi_1-\xi_0)}}}$$

The corresponding frequency is given by

$$f=\frac{kc}{2\pi}=\frac{c}{2\pi h}\quad(kh),$$

where kh is given above.

(*c*) We may refer here to an interesting case, when we take the internal boundary given by $\xi_0=O$. This corresponds to the case (compare with the case in the preceding section) of a membrane of which the line joining the foci remains permanently fixed together with its external elliptic boundary. The frequency corresponding to the case we have considered above is given by

$$f=\frac{kc}{2\pi}=(kh).\frac{c}{2\pi h}=\frac{c}{2\pi h}\cdot\frac{2\sqrt{2}}{\sqrt{\cosh^2\xi_1-\dfrac{2\xi_1\cosh\xi_1}{\sinh\xi_1}}}$$

If we choose the solution

$$z=Se_1(\eta q)\,[ASe_1\,(i\xi,q)+BJn,\,(i\xi,q)]e^{ikct},$$

The frequency is given by

$$f=\frac{kc}{2\pi}=\frac{c}{2\pi h}\;(kh),$$

where

$$kh = \frac{2\sqrt{2}}{\sqrt{\cosh(\xi_0+\xi_1)\cosh(\xi_1-\xi_0) - \frac{2(\xi_1-\xi_0)\sinh\xi_0 \sinh\xi_1}{\sinh(\xi_1-\xi_0)}}}$$

(*d*) When we have an infinitely thin elliptic annulus formed by $\xi=\xi_0$ and $\xi=\xi_1$ so that $\xi_1-\xi_0$ is infinitely small, we have,

$$kh=2\sqrt{2},$$

which shews that, for the order of approximation we have considered the frequency would be the same for all infinitely thin confocal elliptic annuli, formed from the same system of confocal ellipses.

(*e*) Again, if we consider the case of a membrane having the line joining the foci permanently fixed, so that $\xi=0$ is one boundary and $\xi=\xi$, is the outer elliptic boundary, we have,

$$kh=2\sqrt{2}.\ e$$

or $$ka=2\sqrt{2}$$

where a and e are respectively the semi-major axis and eccentricity of the ellipse, whence it follows that for the mode of vibration we are considering, the frequency of vibration is inversely proportional to the major axis and does not depend on the eccentricity, as in the other mode of vibration we have previously considered.

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On Some Hydrodynamical Problems and Associated Legendre Functions and Spherical Harmonics

BY

HRISHIKESH SIRCAR

ON THE MOTION IN AN INFINITE LIQUID, OF AN OBLATE AND A PROLATE SPHEROID ALONG THE COMMON AXIS OF REVOLUTION

Introduction.

The problem of the motion of a system of bodies in an infinite iquid has long attracted the attention of Mathematicians. The problem of two spheres is well-known. Prof. Karl Pearson and others have attempted some problems in connection with spheroids and ellipsoids. The case of a prolate spheroid and oblate spheroid has been considered here, which requires the use of q- and Q-functions. A motion of translation has been contemplated.

1. Preliminary statements.
2. Appropriate transformation formula.
3. Formation of equations.
4. Approximate solution ; the velocity potential.
5. A particular case.

Preliminary Statements.

Problems in which there is a symmetry round the axis of z can often be treated with the aid of a substitution of the form

$$\rho+i\ z=f(\xi+i\ \eta),$$

$(z,\ \rho,\ \omega)$ denoting the ordinary cylindrical co-ordinates of a point in space and $(\xi,\ \eta,\ \omega)$, a system of orthogonal co-ordinates. If $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$, $\nabla^2 V$ can be expressed in terms of the

orthogonal co-ordinates (ξ, μ, ω,) by well-known methods, as

$$\nabla^2 V = h_1 h_2 h_3 \left[\frac{\partial}{\partial \xi}\left(\frac{h_1}{h_2 h_3} \frac{\partial V}{\partial \xi} \right) + \frac{\partial}{\partial n}\left(\frac{h_2}{h_3 h_1} \frac{\partial V}{\partial n} \right) + \frac{\partial}{\partial \omega}\left(\frac{h_3}{h_1 h_2} \frac{\partial V}{\partial \omega} \right) \right],$$

$$\left.\begin{aligned} \text{where } 1/h_1^2 &= \left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 + \left(\frac{\partial z}{\partial \xi} \right)^2 = \frac{\partial(\rho, z)}{\partial(\xi, n)}, \\ 1/h_2^2 &= \left(\frac{\partial x}{\partial n} \right)^2 + \left(\frac{\partial y}{\partial n} \right)^2 + \left(\frac{\partial z}{\partial n} \right)^2 = \frac{\partial(\rho, z)}{\partial(\xi, n)}, \\ 1/h_3^2 &= \left(\frac{\partial x}{\partial \omega} \right)^2 + \left(\frac{\partial y}{\partial \omega} \right)^2 + \left(\frac{\partial z}{\partial \omega} \right)^2 = \rho^2, \end{aligned}\right\} \quad \ldots$$

Since (ρ, z) are conjugate functions of (ξ.n)

and $x=\rho \cos \omega$, $\qquad y=\rho \sin \omega$.

Consequently, Laplace's equation takes the form

$$\frac{\partial}{\partial \xi}\left(\rho \frac{\partial V}{\partial \xi} \right) + \frac{\partial}{\partial n}\left(\rho \frac{\partial V}{\partial n} \right) + \frac{1}{\rho} \frac{\partial(\rho, z)}{\partial(\xi, n)} \quad \frac{\partial^2 V}{\partial \omega^2} = 0$$

For an oblate spheroid the appropriate substitution is

$$\rho + i\, z = a \cosh(\xi + i\, \eta)$$

so that,

$$\left.\begin{aligned} z &= a \sinh \xi \sin \eta = a\mu\zeta \\ \rho &= a \cosh \xi \cos \eta = a\sqrt{1-\mu^2}\sqrt{\zeta^2+1} \end{aligned}\right\} \quad \begin{aligned} \zeta &= \sinh \xi \\ \mu &= \sin \eta. \end{aligned}$$

(ζ, μ, ω) form an orthogonal system and denote the oblate spheroidal co-ordinates of a point. The surfaces $\zeta=$ const. represent the confocal spheroids

$$\frac{\rho^2}{a^2(\zeta^2+1)} + \frac{z^2}{a^2\zeta^2} = 1,$$

and the surfaces $\mu=$ const. represent the confocal hyperboloids of one sheet,

$$\frac{\rho^2}{a^2(1-\mu^2)} - \frac{z^2}{a^2\mu^2} = 1.$$

The element of any arc in any direction λ is

$$ds=\left[\left(\frac{\partial x}{\partial \lambda}\right)^2+\left(\frac{\partial y}{\partial \lambda}\right)^2+\left(\frac{\partial z}{\partial \lambda}\right)^2\right]^{\frac{1}{2}} d\lambda$$

Putting $\lambda=\xi,\ n,\ \omega$ successively, we have,

$$\left.\begin{aligned}
dn_1 &= \sqrt{\frac{\partial(\rho,z)}{\partial(\xi,n)}}\,d\xi=a\sqrt{\frac{\zeta^2+\mu^2}{\zeta^2+1}}\,d\zeta,\\
dn_2 &= \sqrt{\frac{\partial(\rho,z)}{\partial(\xi,\eta)}}\,d\eta=a\sqrt{\frac{\zeta^2+\mu^2}{-\mu^2+1}}\,d\mu,\\
dn_3 &=\rho d\omega \qquad =a\sqrt{1-\mu^2}\,\sqrt{\zeta^2+1}\,d\omega.
\end{aligned}\right\}$$

Laplace's equation is reducible to

$$\frac{\partial}{\partial\zeta}\left[(\zeta^2+1)\frac{\partial V}{\partial\zeta}\right]+\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial V}{\partial\mu}\right]=\left(\frac{1}{\zeta^2+1}-\frac{1}{1-\mu^2}\right)\frac{\partial^2 V}{\partial\omega^2}.$$

In the case of symmetry, this possesses solutions of the form $V=\sum_{n=0}^{\infty} P_n(\mu)[a_n p_n(\zeta)+b_n q_n(\zeta)]$, where a_n, b_n are arbitrary constants and p_n, q_n are harmonic functions defined in Prof. Lamb's Hydrodynamics.

For a prolate spheroid the appropriate substitution is

$$z+i\rho=b\cosh(\xi+i\eta),$$

so that

$$\left.\begin{aligned}
z&=b\cosh\xi\cos n=b\mu_1\zeta_1\\
\rho&=b\sinh\xi\sin\eta=b\sqrt{1-\mu_1^2}\,\sqrt{\zeta_1^2-1}
\end{aligned}\right\}\quad \begin{aligned}\zeta_1&=\cosh\xi\\ \mu_1&=\cos\eta\end{aligned}$$

$(\zeta_1,\ \mu_1,\ \omega)$ form an orthogonal system and denote the prolate spheroidal co-ordinates of a point. The surfaces $\zeta_1=$const. represent the confocal spheroids

$$\frac{x^2+y^2}{b^2(\zeta_1^2-1)}+\frac{z^2}{b^2\zeta_1^2}=1,$$

and the surfaces $\mu=$const. represent the confocal hyperboloids of two sheets,

$$\frac{z^2}{b^2\mu_1^2}-\frac{x^2+y^2}{b^2(1-\mu_1^2)}=1.$$

The elements of arcs, are as before,

$$dn_1 = b\left(\frac{\zeta_1^2-\mu_1^2}{\zeta_1^2-1}\right)^{\frac{1}{2}} d\zeta_1; \quad dn_2 = b\left(\frac{\zeta_1^2-\mu_1^2}{1-\mu_1^2}\right)^{\frac{1}{2}} d\mu_1; \quad dn_3 = \rho d\omega.$$

Laplace's equation is reducible to

$$\frac{\partial}{\partial\zeta_1}\left[(\zeta_1^2-1)\frac{\partial V}{\partial\zeta_1}\right]+\frac{\partial}{\partial\mu_1}\left[(1-\mu_1^2)\frac{\partial V}{\partial\mu_1}\right]+\frac{\zeta_1^2-\zeta_1^2}{(1-\mu_1^2)\ (\zeta_1^2-1)}\frac{\partial^2 V}{\partial\omega^2}=0,$$

which in the case of symmetry, possesses solutions of the type,

$V=\sum\limits_{n=0} [A_n P_n(\zeta_1)+B_n Q_n(\zeta_1)]P_n(\mu_1)$, where A_n B_n are constants and P_n, Q_n are ordinarily Zonal Harmonics.

The spheroids are supposed to have the same axis of revolution ; O and O_1 denote the centres of the oblate and prolate spheroids respectively, c the distance between the centres. (ζ_1, μ_1, ω) (ζ, μ, ω); (z_1, ρ, ω), (z, ρ, ω) represent respectively the prolate spheroidal oblate spheroidal and cylindrical co-ordinates of a point in space, with respect to the two centres as origin. The directions of polar, axes of the spheroids are measured in opposite senses, OO_1 and $O_1 O$, along the common axis of revolution.

We have the geometrical relations in the usual manner,

$$\begin{aligned}\rho &= a\sqrt{\zeta^2+1}\ \sqrt{1-\mu^2} \\ &= b\sqrt{\zeta_1^2-1}\ \sqrt{1-\mu_1^2}\end{aligned} \qquad \text{and } a\mu\zeta+b\mu_1\mu_1=c. \qquad \ldots (1)$$

Let u denote the velocity of O towards O_1 and u_1, that of O_1 towards O, so that c denotes the distance between the two centres at any instant. Let $\zeta=\zeta_0$ and $\zeta_1={}_1\zeta_0$ denote the two given spheroids.

Certain Transformations.

The following transformation fomulae would be useful.

We know the formula, when $\zeta 7 \mu$,

$$\pi P_n(\mu)Q_n(\zeta)=\int_0^\pi Q_n[\mu\zeta+\sqrt{1-\mu^2}\ \sqrt{1-\zeta^2}\ \cos u]du.$$

Its analogue in q-functions for unrestricted real values of ζ, is

$$\pi P_n(\mu) q_n(\zeta = \int_0^\pi p_n[\mu\zeta - i\sqrt{1-\mu^2}\ \sqrt{\zeta^2+1}\ \cos u]du$$

$$= 2^n \sum_{r=0}^{\infty} (-)^r \frac{(n+r)!(n+2r)!}{r!(2n+2r+1)!}$$

$$\int_0^\pi \frac{du}{[\mu\zeta - i\sqrt{1-\mu^2}\ \sqrt{\zeta^2+1}\ cou]^{n+2r+1}}$$

by (1), $= 2^n \sum_{r=0}^{\infty} (-)^r \frac{(n+r)!(n+2r)!a^{n+2r+1}}{r!(2n+2r+1)!}$

$$\int_0^\pi \frac{du}{[c - b\mu_1\zeta_1 - ib\sqrt{1-\mu_1{}^2}\sqrt{\zeta_1{}^2-1}\ \cos u]^{n+2r+1}}$$

$D = \frac{\partial}{\partial c}$, $= \frac{a}{b} 2^n \sum_{r=0}^{\infty} (-)^{n+r} \frac{(n+r)!(aD)^{n+2r}}{r!(2n+2r+1)!}$

$$\int_0^\pi \frac{du}{\frac{c}{b} - [\mu_1\xi_1 - \sqrt{1-\mu_1{}^2}\sqrt{1-\zeta_1{}^2}\cos u]}$$

$c > b$, $= (-)^n \frac{a}{b} \sqrt{\frac{\pi}{2aD}}\, J_{n+\frac{1}{2}}(aD)^{**} \sum_{r=0}^{\infty} (2r+1)Q_r\left(\frac{c}{b}\right)$

$$\int_0^\pi P_r[\mu_1\zeta_1 - \sqrt{1-\mu_1{}^2}\sqrt{1-\zeta_1{}^2}\ \cos u]^* du$$

$$= \frac{\pi a}{b} (-)^n \sqrt{\frac{\pi}{2aD}}\, J_{n+\frac{1}{2}}(aD) \sum_{r=0}^{\infty} (2r+1)Q_r\left(\frac{c}{b}\right) P_r(\mu) P_r(\zeta_1).\dagger$$

Therefore,

$$P_n(\mu) q_n(\zeta) = \sum_{\nu=0}^{\infty} (2r+1),\ {}_1\omega^0{}_n(\gamma, c,\ a,\ b) P_r(\mu_1) P_r(\zeta_1), \qquad \dots \quad (2)$$

** Hydrodynamics, H. Lamb, 5th ed., pp. 478-79.

* Mod. Analysis, Whitaker and Watson, 3rd ed., p. 322, Art. 15.4.

† ,, ,, ,, ,, ,, ,, ,, 328, ,, 15.7.

where

$$ {}_1\omega^0{}_n(r,\ c,\ a.\ b) = (-)^n.\ \frac{a}{b}\ .\ \sqrt{\frac{\pi}{2aD}}\, J^{(aD)}_{n+\frac{1}{2}}\ Qr\left(\frac{c}{b}\right) \quad \ldots \quad (3) $$

Similarly, we may have

$$ P_n(\mu_1)Q_n(\zeta) = \sum_{r=0}^{\infty} (2r-1)\,{}_2\omega^0{}_n(r,\ c,\ b,\ a)P_r(\mu)P_r(\zeta), \quad \ldots \quad (4) $$

where

$$ {}_2\omega^0{}_n(r,\ c,\ b,\ a) = (-)^n\frac{b}{a}\ \sqrt{\frac{\pi}{2bD}}\, J^{(bD)}_{n+\frac{1}{2}}\ q_r\left(\frac{c}{a}\right), \quad \ldots \quad (5) $$

$I_n(z)$ denoting modified Besel-function of the first kind $= j^{-n}J_n(iz)$.**

Equations to determine the Motion.

The velocity potential ϕ of the motion should satisfy the following four conditions:—

(*i*) $\Delta^2\phi = 0$,

(*ii*) $\phi = 0$, at infinity.

(*iii*) $\dfrac{\partial\phi}{\partial\zeta} = -UaP_1(\mu)$, over the oblate spheroid, $\zeta = \zeta_0$,

(*iv*) $\dfrac{\partial\phi}{\partial\zeta_1} = -u_1\ b\ P_1(\mu_1)$, over the prolate spheroid, $\zeta_1 = {}_1\zeta_0$,

Assume

$$ \phi = \sum_{r=0}^{\infty} [A_n P(\mu)q_n(\zeta) + B_n P_n(\mu_1)Q_n(\zeta_1)], $$

which obviously satisfies the conditions (*i*) and (*ii*).

Near the surface of the oblate spheroid, we have, by (4),

$$ \phi = \sum_{r=0}^{\infty} A_r P_r(\mu)q_r(\zeta) + \sum_{n=0}^{\infty} B_n $$

$$ \sum_{r=0}^{\infty} (2r+1)\,{}_2\omega^0{}_n(r,\ c,\ b,\ a)P_r(\mu)p_r(\zeta) \quad \ldots \quad (6) $$

** Mod. Analysis, Whitaker and Watson, 3rd, ed., p. 372, Art. 17.7.

Therefore, the boundary-condition (*iii*) over it gives

$$-u_1 a\mathrm{P}_1 = \sum_{r=0}^{\infty} \mathrm{A}_r \mathrm{P}_r(\mu) q'_r(\zeta)_0 + \sum_{n=0}^{\infty} \mathrm{B}_n$$

$$\sum_{r=0}^{\infty} (2r+1)_2\omega_n{}^0(r.\ c,\ b.\ a)\mathrm{P}_r(\mu) p'_r(\zeta.),$$

where

$$q'_n(\zeta.) = \frac{d}{d\zeta}\, q_n(\zeta),\ \text{for}\ \zeta=\zeta.$$

$$p'_n(\zeta\,) = \frac{d}{d\zeta}\, p_n(\zeta),\ \ ,,\ \ ,,\ \ ,,$$

This being true on the surface of the spheroid, equating the co-effi cients of the various Zonal Harmonics of (μ), we have,

$$\left.\begin{aligned} -ua &= \mathrm{A}_1 q_1(\zeta.) + 3 \sum_{n=0}^{\infty} \mathrm{B}_{n\,2}\omega^0{}_n(1,\ c,\ b,\ a), \\ 0 &= \mathrm{A}_r q'_r(\zeta.) + (2r+1) p'_r(\zeta.) \sum_{n=0}^{\infty} \mathrm{B}_{n\,2}\omega_n{}^0(r,\ c,\ b,\ a),\ r>1 \\ 0 &= \mathrm{A}_0 \end{aligned}\right\} \cdots \text{(A)}$$

Similarly the boundary-condition (*iv*) over the prolate spheroid yields, by (2)

$$\left.\begin{aligned} 0 &= \mathrm{B}_0 \\ -u_1 b &= \mathrm{B}_1 \mathrm{Q}'_1({}_1\zeta_0) + 3 \sum_{n=0}^{\infty} \mathrm{A}_{n\,1}\omega^0{}_n(1,\ c,\ a,\ b), \\ 0 &= \mathrm{B}_r \mathrm{Q}'_r({}_1\zeta_0) + (2r+1)\mathrm{P}'_r(\zeta_0) \sum_{n=0}^{\infty} \mathrm{A}_{n\,1}\omega^0{}_n(r,\ c,\ a,\ b),\ r>1 \end{aligned}\right\} \text{(B)}$$

Substituting from (B) the values of the B's in (A), we get,

$$\left.\begin{aligned} \mathrm{A}_1 - \sum_{n=1}^{\infty} \mathrm{A}_n \theta_{n\,1} &= -\frac{au}{q'_1(\zeta.)} + \frac{3u_1 b}{q'_1(\zeta.)\mathrm{Q}'_1({}_1\zeta.)}\ {}_2\omega_1{}^0(1,\ c,\ b,\ a), \\ \mathrm{A}_r - \sum_{n=1}^{\infty} \mathrm{A}_n \theta_{n\,r} &= (2r+1)\frac{p'_r(\zeta.)}{q'_r(\zeta.)}\ \frac{u_1 b}{\mathrm{Q}'_1({}_1\zeta.)}\ {}_2\omega_1{}^0(r,\ c,\ b,\ a),\ r>1 \end{aligned}\right\} \text{(7)}$$

where

$$\theta nr=(2r+1)\frac{p'_r(\zeta.)}{q'_r(\zeta.)}\sum_{p=1}^{\infty}(2p+1)\frac{P'_p({}_1\zeta.)}{Q'_p({}_1\zeta.)}\,{}_2\omega^{\circ}_p\,(r,c,b,a)\,{}_1\omega^{\circ}_n\,(p,c,a,b) \quad \ldots \quad (8)$$

Similarly, for the determinations of the B's, we get,

$$\left.\begin{aligned} B_1-\sum_{n=1}^{\infty}B_n\phi_{n1} &= \frac{-u_1 b}{Q'_1({}_1\zeta.)}+3\frac{ua}{q'_1(\zeta.)Q'_1({}_1\zeta.)}\,{}_1\omega^{\circ}_1\,(1,c,a,b) \\ B_r-\sum_{n=1}^{\infty}B_n\phi_{nr} &= (2r+1)\frac{P'_r({}_1\zeta.)}{Q'_r({}_1\zeta.)}\cdot\frac{ua}{q'_1(\zeta.)}\,{}_1\omega^{\circ}_1\,(r,c,a,b),\ r>1 \end{aligned}\right\}(9)$$

where

$$\phi_{nr}=(2r+1)\frac{P'_r({}_1\zeta.)}{Q'_r({}_1\zeta.)}\sum_{p=1}^{\infty}(2p+1)\frac{P'_p(\zeta.)}{q'_p(\zeta.)}\,{}_1\omega^{\circ}_p\,(r,c,a,b)\,{}_2\omega^{\circ}_n\,(p,c,b,a) \quad (10)$$

We have an infinity of equations to determine each set of the infinite number of the A's and B's. We, however, proceed to have their values approximately determined.*

Approximate Determinations of the Co-efficients.

We find after some labour, from (5) and (3),

$$\omega^{\circ}_n(r,c,b,a)=\frac{a^r b^{n+1}.2^{n+r}.n!\,r!}{(2n+1)!\,(2r+1)!}\left[\frac{(n+r)!}{C^{n+r+1}}+\frac{(n+r+2)!}{C^{n+r+3}}\left\{\frac{b^2}{2(2n+3)}-\frac{a^2}{2(2n+3)}\right\}+\ \ldots\ldots\right] \ldots\ldots(11)$$

$$\omega^{\circ}_n(r,c,a,b)=\frac{a^{n+1}b^2.n!\,r!\,2^{n+r}}{(2n+1)!\,(2r+1)!}\left[\frac{(n+r)!}{C^{n+r+1}}+\frac{(n+r+2)!}{C^{n+r+3}}\left\{\frac{b^2}{2(2r+3)}-\frac{a^2}{2(2n+3)}\right\}+\ \ldots\ldots\right] \ldots\ldots\ldots (12)$$

$$\theta nr=(2r+1)\frac{p'_r(\zeta.)}{q'_r(\zeta.)}\sum_{p=1}^{\infty}(2p+1)\frac{P'_p({}_1\zeta.)}{Q'_p({}_1\zeta.)}$$

* The co-efficients of A's and B's are supposed to be expanded in positive integral powers of $\frac{\text{Linear dimensions}}{\text{Central distance}}$ on physical grounds.

$$\frac{a^{n+r+1}.b^{2p+1}.2^{n+r+2p}n\,!r\,!p\,!p\,!}{(2n+1)\,!(2p+1)\,!(2p+1)\,!(2r+1)\,!}\left[\frac{(p+r)\,!(p+n)\,!}{C^{n+r+2p+2}}+\ \cdots\right]\cdots \quad (13)$$

$$\phi nr=(2r+1)\,\frac{P'_r({}_1\zeta.)}{Q'_r({}_1\zeta.)}\sum_{p=1}^{\infty}(2p+1)\,\frac{P'_p({}_1\zeta.)}{q'_p({}_1\zeta.)}.$$

$$\frac{a^{2p+1}.b^{n+r+1}.2^{n+r+2p}n\,!r\,!p\,!p\,!}{(2n+1)\,!(2p+1)\,!(2p+1)(2r+1)\,!}\left[\frac{(p+r)\,!(p+n)\,!}{C^{n+r+2p+2}}+\ \cdots\cdots\right]\cdots (14)$$

If the central distance be such that we can neglect terms of order $\left(\dfrac{\text{Linear dimensions}}{\text{Central distance}}\right)^3$ and higher, we get,

$$\left.\begin{array}{l} A_1 = -\dfrac{au}{q'_1(\zeta.)}\ ;\quad A_r=0,\ r>1 \\[2ex] B_1 = \dfrac{-u_1 b}{Q'_1({}_1\zeta.)}\ ;\quad B_r=0,\ r>1 \end{array}\right\}\quad \cdots\quad \cdots\quad (15)$$

The corresponding value of ϕ is

$$\phi=A_1P_1(\mu)q_1(\zeta)+B_1P_1(\mu_1)Q_1(\zeta_1).\quad \cdots\quad (16)$$

(*ii*) If the central distance be such that terms of order $\left(\dfrac{\text{Linear dimensions}}{\text{Central distance}}\right)^5$ and higher may be neglected, we get.

$$A_1=-\frac{au}{q'_1(\zeta.)}+\frac{2}{3}\,\frac{u_1b}{q'_1(\zeta.)Q'_1({}_1\zeta.)}\,\frac{ab^2}{c^3}\ ;\ A_2=\frac{2}{3}\,\frac{P'^2(\zeta.)}{q'_2(\zeta.)}\cdot\frac{u_1b}{(Q'_1\zeta.)}\cdot\frac{a^2b^2}{c^4}\ ;$$

$$A_r=0,\quad r>2\ ;$$

$$B_1=\frac{-u_1b}{Q'_1({}_1\zeta.)}+\frac{2}{3}\,\frac{ua}{q'_1(\zeta.)Q'_1({}_1\zeta.)}\,\frac{a^2b}{c^3}\ ;\ B_2=\frac{2}{3}\,\frac{P'_2({}_1\zeta.)}{Q'_2({}_1\zeta.)}\,\frac{ua}{q'_1(\zeta.)}\cdot\frac{a^2b^2}{c^4}$$

$$B_r=0,\quad r>2\ :$$

The corresponding value of ϕ is

$$\phi=A_1P_1(\mu)q_1(\zeta)+A_2P_2(\mu)q_2(\zeta)+B_1P_1(\mu_1)Q_1(\gamma_1)+B_2P_2(\mu)Q_2(\zeta_1)\ (17)$$

(*iii*) If the central distance be such that terms of order $\left(\dfrac{\text{Linear dimensions}}{\text{Central distance}}\right)^7$ and higher may be ignored, we get.

$$A_1 = -\frac{au}{q'_1(\zeta_\cdot)}\left[1+\tfrac{4}{5}\frac{a^3b^3}{c^6}\cdot\frac{1}{q'_1(\zeta_\cdot)Q'_1({}_1\zeta_\cdot)}\right]+\tfrac{2}{3}\,\frac{u_1 b}{q'_1(\zeta_\cdot)Q'_1({}_1\zeta_\cdot)}$$

$$\left[\frac{ab^2}{c^3}+\tfrac{6}{5}\,\frac{ab^2}{c^5}\,(b^2-a^2)\right]$$

$$A_2 = \frac{p'_2(\zeta_\cdot)}{q'_2(\zeta_\cdot)}\cdot\frac{u_1 b}{Q'_1({}_1\zeta_\cdot)}\left[\frac{a^2b^2}{c^4}+2\,\frac{a^2b^2}{c^6}(b^2-\tfrac{5}{7}a^2)\right],$$

$$A_3 = \tfrac{8}{15}\,\frac{p'_3(\zeta_\cdot)}{q'_3(\zeta_\cdot)}\cdot\frac{u_1 b}{Q'_1({}_1\zeta_\cdot)}\cdot\frac{a^3b^2}{c^5},$$

$$A_4 = \tfrac{8}{21}\,\frac{p'_4(\zeta_\cdot)}{q'_4(\zeta_\cdot)}\cdot\frac{u_1 b}{Q'_1({}_1\zeta_\cdot)}\cdot\frac{a^4b^2}{c^6}.$$

$A_r = 0 \qquad r > 4.$

$$B_1 = -\frac{u_1 b}{Q'_1({}_1\zeta_\cdot)}\left[1+\tfrac{4}{5}\frac{a^3b^3}{c^6}\cdot\frac{1}{q'(\zeta_\cdot)Q'_1({}_1\zeta_\cdot)}\right]+$$

$$\tfrac{2}{3}\,\frac{ua}{q'_1(\zeta_\cdot)Q'_1({}_1\zeta_\cdot)}\left[\frac{a^2b}{c^3}+\tfrac{6}{5}\frac{6a^2b}{c^5}\,(b^2-a^2\right],$$

$$B_2 = \frac{2}{3}\,\frac{P'_2({}_1\zeta_\cdot)}{Q'_2({}_1\zeta_\cdot)}\cdot\frac{ua}{q_1'(\zeta_\cdot}\quad \frac{a^2b^2}{}\quad \frac{2a^2b^2}{c^6}\left(\frac{5}{7}b^2-a^2\right)\Big]$$

$$B_3 = \frac{8}{15}\,\frac{P'_3({}_1\zeta_\cdot)}{Q'_3({}_1\zeta_\cdot)}\cdot\frac{ua}{q'_1(\zeta_\cdot)}\cdot\frac{a^2b^3}{c^5},$$

$$B_4 = \frac{8}{21}\,\frac{P'_4({}_1\zeta_\cdot)}{Q'_4({}_1\zeta_\cdot)}\cdot\frac{ua}{q'_1(\zeta_\cdot)}\cdot\frac{a^2b^4}{c^6},$$

$B_r = 0 \qquad r > 4.$

A Particular Case.

The case of a sphere and spheroid is obvious. If, however, we put $\zeta = 0$, the oblate spheroid reduces to a circular disc of radius a, so that we would get the motion corresponding to a circular disc and a prolate spheroid in an infinite liquid. Since* $p'_{2n}(o) = 0$, $\dfrac{1}{q'_1(o)} = -\dfrac{2}{\pi}$, and $\dfrac{p'_{2p+1}(o)}{q'_{2p+1}(o)} = -\dfrac{2}{\pi}$, the even A's are all zero and we can write down readily the corresponding approximate value of other co-efficients.

* Phil. Trans. Royal Soc., Vol. 224, 1924, p. 53. Dr. Nicholson, "On Spheroidal Harmonics, etc."

Motion in an Infinite Liquid due to the Translation of Two Equal Co-axial Parallel Circular Discs along the Common Axis

Introduction.

The motion of translation of a single circular disc * in an infinite liquid is well-known, the translation of two circular discs has been considered in the present paper. The method followed is that of Dr. J. W. Nicholson † in his Memoir on the Electrification of Two Circular Discs. Fourier-Bessel Integral of spheroidal Harmonics and a transformation have been necessary. An Integral equation has been found, of Dr. Nicholson's type, which has been approximately solved. Dr. Nicholson's Memoir alluded to above, has been of considerable help to me during the preparation of this paper.

1. Preliminary statements.
2. Appropriate transformation formulae.
3. Equations.
4. Formation of integral equations.
5. Approximate solution.
6. Fourier-Bessel Integral.
7. The velocity potential.

Preliminary Statements.

Two oblate spheroids of different radii of confocality, a, b, are contemplated, having the same axis of revolution which is taken as the z-axis; (ζ, μ, ω) and (ζ_1, μ_1, ω) denote the oblate spheroidal co-ordinates of a point in space with different origins o (the centre of the oblate spheroid, b) and o_1 (the centre of the oblate spheroid, a) respectively ; $(z_1\, \rho. \omega)$ and $(z. \rho, \omega)$ denote the corresponding cylindrical co-ordinates, the Z-axis being measured positively in

* Hydrodynamics, H. Lamb, 5th ed., p. 135.

† Phil. Trans. Royal Soc., Vol 224, 1924.

opposite senses, oo_1 and o_1o, respectively. We then have the geometrical relations,

$$z+z'=c,\ oo_1=c,$$

$$\text{and } \rho=a\sqrt{1-\mu^2}\ \sqrt{\zeta_1{}^2+1}=b\sqrt{1-\mu^2}\ \sqrt{\zeta^2+1}, \qquad \ldots \quad (1)$$

i.e., $a\mu_1\zeta_1+b\mu\zeta=c$

The spheroids are supposed to move with velocities V_1 and V_2 along the common axis of revolution.

Appropriate Transformation Formulae.

Deriving the analogue in q-function (those introduced in Prof. Lamb's Hydrodynamics) of a well-known formula in Q-functions, we have, as in the foregoing problem,

$$P_n(\mu_1)q_n(\zeta_1)=\sum_0^\infty (2r+1)\,{}_1\omega_n{}^o(\nu,c,a,b)P_r(\mu)\mathfrak{p}_r(\zeta), \qquad \ldots \quad (2)$$

where

$${}_1\omega_n{}^o(\gamma,c,ab)=(-)^n\frac{a}{b}\sqrt{\frac{\lambda}{2aD}}\ J^{(aD)}_{n+\frac{1}{2}}\ q_r{}^3\left(\frac{c}{a}\right),\quad D=\frac{\partial}{\partial c}$$

$$=(-)^n\sqrt{\frac{a}{b}}\cdot\frac{\Lambda}{2}\cdot\frac{J_{n+\frac{1}{2}}(aD)}{\sqrt{D}}\int_0^\infty e^{-\lambda c}\ J^{(\lambda b)}_{r-\frac{1}{2}}\ \frac{d\lambda}{\sqrt{\lambda}}$$

$$=\frac{\pi}{2}\sqrt{\frac{a}{b}}\int_0^\infty e^{-\lambda c}\ J^{(\lambda b)}_{r+\frac{1}{2}}\ J^{(\lambda a)}_{n+\frac{1}{2}}\ \frac{d\lambda}{\lambda} \qquad \ldots \quad (3)$$

Similarly,

$$P_n(\mu)q_n(\zeta)=\sum_{r=0}^\infty (2r+1)\,{}_2\omega_n^o(\nu_1c_1b_1a)\ P_r(\mu_1)\mathfrak{p}_r(\zeta_1), \qquad \ldots \quad (4)$$

where

$${}_2\omega_n^o(\gamma,c,b.\,a)=(-)^n\frac{b}{a}\sqrt{\frac{\lambda}{2bD}}\ J^{(bD)}_{n+\frac{1}{2}}\ q_r\left(\frac{c}{a}\right)$$

$$=(-)^n\frac{\pi}{2}\sqrt{\frac{b}{a}}\int_0^\infty J^{(\lambda a)}_{r+\frac{1}{2}}\cdot\frac{J^{(Db)}_{n+\frac{1}{2}}}{\sqrt{D}}\cdot e^{-\lambda c}\cdot\frac{d\lambda}{\sqrt{\lambda}}$$

$$=\frac{\lambda}{2}\sqrt{b/a}\int_0^\infty J^{(\lambda a)}_{r+\frac{1}{2}}\ J^{(\lambda b)}_{n+\frac{1}{2}},e^{-\lambda c}\,\frac{d\lambda}{\lambda}. \qquad \ldots \quad (5)$$

Equations to determine Co-efficients in the Velocity Potential.

The velocity-potential of the motion should satisfy the following conditions:—

(*i*) $\Delta^2\phi=0$

(*ii*) $\phi=0$, at infinity.

(*iii*) $\dfrac{\partial\phi}{\partial\zeta} = -V_1\, bP_1(\mu)$, over the spheroid. b, $\zeta=\zeta_o$,

(*iv*) $\dfrac{\partial\phi}{\partial\zeta_1}=-V_2 aP_1(\mu_1)$,, ,, ,, a, $\xi,=,\zeta_o$.

Assume

$$\phi=\sum_{n=o}^{\infty}\left[b_n P_n(\mu)q_n(\zeta)+a_n P_n(\mu)q_n(\zeta_1)\right] \quad \dots \quad (6)$$

Near the surface of the spheroid, b, we have, by (2),

$$\phi=\sum^{\infty} b_r P_r(\mu)q_r(\zeta)+\sum_{n=o}^{\infty}a_n\sum_{r=o}^{\infty}(2r+1)\,{}_1\omega_n{}^{o}(\nu,c,a.b)\ P_r(\mu)\mathfrak{p}_r(\zeta).$$

Therefore, the boundary-condition over it gives,

$$-bv_1P_1(\mu)=\sum_{r=o} b_r P_r(\mu)q_r{}'(\zeta_o)+\sum_{n=o}^{\infty}a_n\sum_{r=o}^{\infty}(2\gamma+1)\,{}_1\omega_n{}^{o}(\nu,c,a,b)\ P_r(\mu)\mathfrak{p}_r{}'(\zeta_0)$$

whence, equating the co-efficients of the Zonal Harmonics, we have,

$$o=b.$$

$$\frac{-bV}{q'_1(\zeta)}=b_1+3\frac{\mathfrak{p}'_1(\zeta.)}{q'_1(\zeta.)}\sum_{n=o}^{\infty}a_n\,{}_1\omega_n{}^{o}(1,\ c,\ a,\ b).$$

$$o=b\gamma+(2\nu+1)\frac{\mathfrak{p}'_r(\zeta.)}{q'_r(\zeta.)}\sum_{n=o}^{\infty}a_n 1\omega_n{}^{o}\ (\gamma,\ c,\ a,\ b);\ \gamma>1 \quad \dots \quad (7)$$

Similarly, the boundary-condition over the other spheroid yields, on using (4),

$$\left.\begin{aligned} &o=a \\ &-\frac{aV_2}{q_{\iota 1}(\nu'.)}=a_1+3\frac{p'_1(\xi'.)}{q'_1(\xi'.)}\sum_{n=o}^{\infty}b_n\ {}_2\omega_n^{o}\ (1,\ c.\ a,\ b) \\ &o=a_r+(2r+1)\frac{p'_r(\zeta'.)}{q'_r(\zeta'.)}\sum_{n=o}^{\infty}bn\ {}_2\omega_n{}^{o}(r,\ c,\ a,\ b);\ r>1 \end{aligned}\right\} \quad \dots \quad (9)$$

Two Parallel Co-axial Circular Discs.

In the case of two circular discs, we have, $\zeta_\circ = \zeta'_\circ = 0$, and since $\left(\frac{d}{d\zeta} p_{2n}\right) = 0$, $\frac{p'_{2n+1}(o)}{q'_{2n+1}(o)} = -\frac{2}{\pi}$, and $\frac{1}{q'_1(o)} = -\frac{2}{\pi}$, we see that the even co-efficients a_{2n}, b_{2n} vanish in the expression for the velocity-potential, and the above conditions reduce to,

$$\left.\begin{aligned} b_1 &= \frac{2b}{\pi} V_1 + 3.\frac{2}{\pi}\sum_{n=0}^{\infty} a_{2n+1}\ {}_1\omega^{\circ}_{2n+1}(1, c, a, b). \\ b_{2r+1} &= \frac{2}{\pi}(4r+3)\sum_{n=0}^{\infty} a_{2n+1}\ {}_1\omega^{\circ}_{2n+1}(2r+1, c, a, b),\ r>0 \end{aligned}\right\} \dots (9)$$

and
$$\left.\begin{aligned} a_1 &= \frac{2aV_2}{\pi} + 3.\frac{2}{\pi}\sum_{n=0}^{\infty} b_{2n+1}\ {}_2\omega^{\circ}_{2n+1}(1, c, a, b) \\ a_{2r+1} &= \frac{2}{\pi}(4r+3)\sum_{n=0}^{\infty} b_{2n+1}\ {}_2\omega^{\circ}_{2n+1}(2r+1, c, a, b),\ r>0 \end{aligned}\right\} \dots (10)$$

Fourier-Bessel Integral of Spheroidal Harmonics.

$$P_n(\mu)q_n(\zeta) \equiv \frac{1}{\pi}\int_0^{\pi} q_n[\mu\zeta - 1\sqrt{1-\mu^2}\sqrt{\zeta^2+1}\cos u]du$$

$$= \sqrt{\frac{\pi b}{2}}\cdot\frac{1}{\pi}\cdot\int_0^{\pi}\int_0^{\infty} J_{n+\frac{1}{2}}(\lambda b)\frac{d\lambda}{\sqrt{\lambda}}\, e^{-\lambda[b\mu\zeta - ib\sqrt{1-\mu^2}\sqrt{\zeta^2+1}\cos u]} du$$

by a change of order,

$$= \frac{1}{\pi}\sqrt{\frac{\pi b}{2}}\int_0^{\infty} e^{-\lambda z} J_{n+\frac{1}{2}}(\lambda b)\frac{d\lambda}{\sqrt{\lambda}}\int_0^{\pi} e^{i\lambda\rho\cos u} du.$$

$$= \frac{1}{\pi}\sqrt{\frac{\pi b}{2}}\int_0^{\infty} e^{-\lambda z} J_{n+\frac{1}{2}}(\lambda b)\frac{d\lambda}{\sqrt{\lambda}}$$

$$\int_0^{\pi}[J_0(\lambda\rho) + 2\sum_1^{\infty} i^s J_s(\lambda\rho)\cos us]^* du$$

† *Loc. cit.* or, Nicholson's Paper.

* The Theory of Bessel Functions, G. N. Watson, p. 22, Arts. 2-22.

$$= \sqrt{\frac{\pi b}{2}} \int_0^\infty e^{-\lambda z} J_{n+\frac{1}{2}}(\lambda b) J_0(\lambda \rho) \frac{d\lambda}{\sqrt{\lambda}} \qquad \dots \quad (11)$$

or differently thus,

$$P_n^m(\mu) q_n^m(\zeta) = \frac{i^m}{\pi} \frac{(n+m)!}{(n-m)!} \int_0^\pi q_n[\mu\zeta - i\sqrt{1-\mu^2}\sqrt{\zeta^2+1}\cos u]$$

$$\cos mu\, du = \frac{i^m}{\pi} \frac{(n+m)!}{(n-m)!} \sum_0^\infty (-)^r 2^n \frac{(n+r)!(n+2r)!}{r!(2n+2r+1)!}$$

$$\int_0^\pi \frac{\cos mudu\, b^{n+2r+1}}{(z-i\rho\cos u)^{n+2r+1}}$$

introducing polar co-ordinates for z, ρ,

$$= \frac{i^m}{\pi} \frac{(n+m)!}{(n-m)!} \sum_0^\infty (-)^r 2^n \frac{(n+r)!(n+2r)!}{r!(2n+2r+1)!} \left(\frac{b}{R}\right)^{n+2r+1}$$

$$\int_0^\pi \frac{\cos mudu}{[\mu' + \sqrt{\mu'^2-1}\cos u]^{n+2r+1}}$$

$$= (-)^m \frac{(n+m)!}{(n-m)!} \sum_0^\infty (-)^r 2^n \frac{(n+r)!(n+2r-m)!b^{n+2r+1}}{r!(2n+2r+1)!}$$

$$P^m_{n+2r}/R^{n+2r+1}$$

$$= (-)^m \frac{(n+m)!}{(n-m)!} \int_0^\infty e^{-\lambda z} J_m(\lambda b) J_{n+\frac{1}{2}}(\lambda b) \frac{d\lambda}{\sqrt{\lambda}} \cdot \sqrt{\frac{\pi b}{2}}$$

using Prof. Hobson's formula.

$$P_n^m(\mu)/r^{n+1} = \frac{1}{\sqrt{n-m+1}} \int_0^\infty e^{-\lambda z} J_m(\lambda\rho)\lambda^n d\lambda,$$

and the form in Series of $r^{-\frac{1}{2}} J^{(r)}_{n+\frac{1}{2}}$, as given in Prof. Lamb's Hydrodynamics, pp. 478-79 ... (12)

The above is only a different way of obtaining Dr. Nicholson's result; his method as applied to spheroidal Harmonics of zonal types seems to be unsuitable when spheroidal Harmonics of associated types are concerned; but the above methods answer all purposes.

Two Simultaneous Integral Equations.

Since $${}_1\omega^0{}_{2n+1}(2r+1,\ c,\ a,\ b)=+\frac{\pi}{2}\sqrt{\frac{a}{b}}$$

$$\int_0^\infty e^{-cx} J^{(ax)}_{2n+\frac{3}{2}}\, J^{(bx)}_{2r+\frac{3}{2}}\, \frac{dx}{x}\ ,\ [\text{see } 3)]$$

and

$${}_2\omega^0{}_{2n+2}(2r+1,\ c.\ a,\ b)=+\frac{\pi}{2}\sqrt{\frac{b}{a}}$$

$$\int_0^\infty e^{-cx}\, J^{(ax)}_{2r+\frac{3}{2}}\, J^{(xb)}_{2n+\frac{3}{2}}\, \frac{dx}{x}\ ,\ [\text{see } (5)]$$

we have, from (10),

$$a_{2r+1}=\frac{2}{\pi}(4r+3)\sum_{n=0}^{\infty} b_{2n+1}\ {}_2\omega^0{}_{2n+1}(2r+1.\ c,\ a,\ b),$$

$$\text{by } (5),=+(4r+3)\sqrt{\frac{b}{a}}\sum_{n=0}^{\infty}\int_0^\infty e^{-ca}\, J^{(xa)}_{2r+\frac{3}{2}}\, J^{(xb)}_{2u+\frac{3}{2}}\, b_{2n+1}\frac{dx}{x}\ .$$

$$=+\sqrt{\frac{b}{a}}\int_0^\infty e^{-cx}\,\frac{dx}{x}\ .\ (4r+3)\ J^{(xa)}_{2r+\frac{3}{2}}\sum_0^{\infty} b_{2n+1}\, J^{(xb)}_{2n+\frac{3}{2}}$$

$$=+\sqrt{\frac{b}{a}}\int_0^\infty e^{-cx}\,\frac{dx}{x\sqrt{a}}\ F(xb)(4r+3)\, J^{(xa)}_{2r+\frac{3}{2}}\ . \qquad \ldots \quad (13)$$

$$\text{where } F(\lambda b)=\sqrt{a}\sum_0^{\infty} b_{2n+1}\, J^{(xb)}_{u+\frac{3}{2}}\ . \qquad \ldots \quad (13a)$$

Similarly, from (9) and (3)

$$b_{2r+1}=+\sqrt{\frac{a}{b}}\int_0^\infty e^{-cx}\,\frac{dx}{x\sqrt{b}}\, f(xa)(4r+3)\, J^{(xa)}_{2n+\frac{3}{2}}\ , \qquad \ldots \quad (14)$$

$$\text{where } f(\lambda a)=\sqrt{b}\sum_0^{\infty} a_{2n+1}\, J^{(xa)}_{2n+\frac{3}{2}} \qquad \ldots \quad (14a)$$

except when $r=0$, in which case, we must add $\frac{2av_2}{\pi}$ and $\frac{2bv_1}{\pi}$ on the right hand sides respectively, of (13) and (14).

Let us now introduce another variable y, independent of x, and multiply the above equations by $J^{(ay)}_{2r+\frac{3}{2}}$ and $J^{(by)}_{2r+\frac{3}{2}}$ respectively; then summing for all positive integral values of r, we have

$$\left.\begin{aligned} &\sum_0^\infty a_{2r+1} J^{(ay)}_{2r+\frac{3}{2}} = \frac{2av_2}{\pi} J^{(ay)}_{\frac{3}{2}} + \sqrt{\frac{b}{a^2}} \\ &\int_0^\infty e^{-cx} \frac{dx}{x} F(bx) \sum_0^\infty (4r+3) J^{(xa)}_{2r+\frac{3}{2}} J^{(ay)}_{2r+\frac{3}{2}} \\ \text{and}\quad &\sum_0^\infty b_{2r+1} J^{(by)}_{2r+\frac{3}{2}} = \frac{2bv_1}{\pi} J^{(ay)}_{\frac{3}{2}} + \sqrt{\frac{a}{b^2}} \\ &\int_0^\infty e^{-cx} \frac{dx}{x} f(xa) \sum_0^\infty (4r+3) J^{(xb)}_{2r+\frac{3}{2}} J^{(yb)}_{2r+\frac{3}{2}}\ . \end{aligned}\right\} \quad \ldots \quad (15)$$

But we know from a special case of Gegenbaner's theorem,*

$$\sum_0^\infty (-)^s (2s+1)\, J^{(x)}_{s+\frac{1}{2}}\, J^{(y)}_{s+\frac{1}{2}} = \frac{2}{\pi}\sqrt{xy}\,\frac{\sin(x+y)}{x+y},\ x \neq y$$

and also

$$\sum_0^\infty (2s+1)\, J^{(x)}_{s+\frac{1}{2}}\, J^{(y)}_{s+\frac{1}{2}} = \frac{2}{\pi}\sqrt{xy}\,\frac{\sin(x-y)}{x-y},$$

whence by subtraction

$$\sum_0^\infty (4r+3)\, J^{(x)}_{2r+\frac{3}{2}}\, J^{(y)}_{2r+\frac{3}{2}} = \frac{\sqrt{xy}}{\pi}\left[\frac{\sin(x-y)}{x-y} - \frac{\sin(x+y)}{x+y}\right] \quad \ldots \quad (16)$$

Hence, from (15),

$$\frac{f(ay)}{\sqrt{by}} - \frac{2av_2}{\pi}\,\frac{J^{(ya)}_{\frac{3}{2}}}{\sqrt{y}} = +\frac{1}{\pi}\sqrt{\frac{b}{a^2}}\int^\infty e^{-cx}\frac{dx}{\sqrt{x}} F(xb)$$

$$\left[\frac{\sin a(x-y)}{x-y} - \frac{\sin a(x+y)}{x+y}\right] \quad \ldots \quad \ldots \quad (17)$$

* Math. Ann., 11, 1871; the Theory of Bessel Functions, p. 525, Watson.

and from (15),

$$\frac{F(yb)}{\sqrt{ya}} - \frac{2bv_1}{\pi} \cdot \frac{J_{\frac{3}{2}}(yb)}{\sqrt{y}} = +\frac{1}{\pi}\sqrt{\frac{a}{b^2}}\int_0^{\infty} e^{-cx}\frac{dx}{\sqrt{x}}\, f(xa)$$

$$\left[\frac{\sin b(x-y)}{x-y)} - \frac{\sin b(x+y)}{x+y}\right], \quad \ldots \qquad \ldots \quad (18)$$

which are the two simultaneous Integral equations.

Two Equal Co-axial Parallel Circular Discs.

In the case of two equal circular discs, $a=b$, and the above two Integral equations reduce to,

$$\frac{f(ya)}{\sqrt{ya}} - \frac{2av_2}{\pi}\frac{J_{\frac{3}{2}}(ya)}{\sqrt{y}} = +\frac{1}{\pi}\int_0^{\infty} e^{-cx}\frac{dx}{\sqrt{xa}}\, F(xa)$$

$$\left[\frac{\sin a(x-y)}{x-y} - \frac{\sin a(x+y)}{x+y}\right] \quad \ldots \qquad \ldots \quad (19)$$

and
$$\frac{F(ya)}{\sqrt{ya}} - \frac{2av_1}{\pi}\frac{J_{\frac{3}{2}}(ya)}{\sqrt{y}} = +\frac{1}{\pi}\int_0^{\infty} e^{-cx} \cdot \frac{dx}{\sqrt{xa}}\, f(xa)$$

$$\left[\frac{\sin a(x-y)}{x-y} - \frac{\sin a(x+y)}{x+y}\right] \quad \ldots \qquad \ldots \quad (20)$$

Two Special Cases.

(*i*) If $v_1 = v_2$, we have by subtraction, from (19) and (20)

$$\frac{f(ya)}{\sqrt{ya}} - \frac{F(ya)}{\sqrt{ya}} = -\frac{1}{\pi}\int_0^{\infty} e^{-xc}\, dx. \left[\frac{f(xa)}{\sqrt{xa}} - \frac{F(xa)}{\sqrt{xa}}\right]$$

$$\left[\frac{\sin a(x-y)}{x-y} - \frac{\sin a(x+y)}{x+y}\right] \quad \ldots \qquad \ldots \quad (21)$$

This is a homogeneous integral equation of which the only continuous solution is zero,* so that we have $f=F$ and consequently,

$$\frac{f(ya)}{\sqrt{ya}} - \frac{2a\nu_2 J_{\frac{3}{2}}(ya)}{\sqrt{y}} = \frac{1}{\pi}\int_0^\infty e^{-xc}\frac{dx}{\sqrt{xa}} f(xa)$$

$$\left[\frac{\sin a(x-y)}{x+y} - \frac{\sin a(a+y)}{x+y}\right] \qquad \ldots \quad (\alpha)$$

(*ii*) If $\nu_1 = -\nu_2$, we have by addition and arguing as before, $f=-F$, whence

$$\frac{f(ya)}{\sqrt{ya}} - \frac{2a\nu_2}{\pi}\cdot\frac{J_{\frac{3}{2}}(ya)}{\sqrt{y}} = -\frac{1}{\pi}\int_0^\infty e^{-xc}\frac{dx}{\sqrt{xa}} f(xa)$$

$$\left[\frac{\sin a(x-y)}{x-y} - \frac{\sin a(x+y)}{x+y}\right] \quad \ldots \qquad \ldots \quad (\beta)$$

Approximate Solutions of the Integral Equations (α) *and* (β).

The same method of approximation may be applied both to (α) and (β). Let us consider the Integral equation (β).

Putting $z=ya$, the equation (β) reduces to

$$\frac{F(z)}{\sqrt{z}} - \frac{2\nu_2 a^{\frac{3}{2}}}{\pi}\frac{J(z)}{\sqrt{z}} = -\frac{1}{\pi}\int_0^\infty e^{-\frac{xc}{a}}\frac{dx}{\sqrt{\ }} F(x)$$

$$\left[\frac{\sin(x-z)}{x-x} - \frac{\sin(x+z)}{x+z}\right]$$

* Mod. Analysis, Whittaker and Watson. 3rd ed., p. 217. From (9) and (10) it may be easily seen, in the case of equal circles, that $a_{2n+1}=b_{2n+1}$ and consequently from (13*a*) and (14*a*) we have $f=F$, which corroborates the validity of the conclusion and indicates that $-\frac{1}{\pi}$ is not a root of the equation $D(\lambda)=0$, of the homogeneous integral equation.

Let

$$I=\int_0^\infty e^{-kx}dx\left[\frac{\sin(x-z)}{(x-z)}-\frac{\sin(x+z)}{x+z}\right],\quad k=c/a.$$

$$=\int_0^\infty a^{-kx}dx\int_0^1 da\ [\cos a(x-z)-\cos a(x+z)],$$

$$=2\int_0^1 \sin azda\int_0^{} e^{--}\sin\ axdx,$$ [by obviously valid

inversion of order]

$$=2\int_0^1 \frac{\sin az.ada}{a^2+k^2}.$$

$$=2\sum_{r=0}^\infty(-)^r.\frac{z^{2r+1}}{(2r+1)!}\int_0^1\frac{a^{2r+2}}{a^2+k^2}da.$$

Therefore

$$D^nI=(-)^n\int x^n e\left[\frac{\sin(x-z)}{x-z}-\frac{\sin(x+z)}{x+z}\right],\ D=\frac{\partial}{\partial\kappa}=a\frac{\partial}{\partial c}$$

$$=2\sum_{r=0}^\infty(-)^r\cdot\frac{z^{2r+1}}{(2r+1)!}D^n\int_0^1\frac{a^{2r+2}}{a^2+k^2}da. \quad\ldots\quad (22)$$

Now let

$\frac{f(z)}{\sqrt{z}}=\sum_0 B_n z^n$. Then the above Integral equation becomes by using (22)

$$\sum_0^\infty B_nz^n-\left(\frac{2a}{\pi}\right)^{\frac{3}{2}}V_0\sum_{n=0}^\infty(-)^n.\frac{2n+2}{(2n+3)!}.z^{2n+1}$$

$$=-\frac{2}{\pi}\sum_{n=0}^\infty\frac{(-)^n}{B_n}\sum_{r=0}^\infty(-)^\nu\frac{z^{2\nu+1}}{(2\nu+1)!}D^n\int_0^1\frac{a^{2r+2}}{a^2+k^2}da,$$

from which we observe that only odd powers of z occur in $\frac{f(z)}{\sqrt{z}}$ and accordingly, we have,

$$B_{2\nu+1}-\left(\frac{2a}{\pi}\right)^{\frac{3}{2}}V_2\,(-)^{\gamma}.\frac{2\nu+2}{(2\nu+3)\,!}=\frac{2}{\pi}\;\frac{(-)^{\nu}}{(2\nu+1\,!}\;\sum_{n=0}^{\infty}$$

$$B_{2n+1}\,D^{2n+1}$$

$$\int_0^1\frac{a^{2\nu+2}}{a^2+k^2}\,da.=(-)^{\nu+1}.\frac{2}{\pi}.\frac{1}{(2\nu+1)\,!}\;\sum_{n=0}^{\infty}B_{2n+1}$$

$$\left[\frac{(2n+2)\,!}{(2\nu+3)k^{2n+3}}-\frac{(2n+4)\,!}{3\,!\,(2\nu+5)k^{2n+5}}+\frac{(2n+6)\,!}{5\,!\,(2\nu+7)k^{2n+7}}\cdots\cdots\right] \tag{23}$$

Approximation.

(*i*) Central distance c being such that only the third power of $\frac{a}{c}$ may be retained, we have,

$$B_{2\nu+1}-\left(\frac{2a}{\pi}\right)^{\frac{3}{2}}V_2\,(-)^{\gamma}\,\frac{2\gamma+2}{(2\gamma+3)\,!}=(-)^{\gamma+1}.\frac{4}{\pi}\,\frac{B_1}{(2\gamma+1)\,!}\cdot\frac{1}{2\gamma+3}$$

$$.\left(\frac{a}{c}\right)^3;$$

Putting $\nu=0$, we have,

$$B_1=\left(\frac{2a}{\pi}\right)^{\frac{3}{2}}\frac{V_2}{3}\left[1-\frac{4}{3\pi}\left(\frac{a}{c}\right)^3\right],\text{ and}$$

$$B_{2\gamma+1}=(-)^{\gamma}.\left(\frac{2a}{\pi}\right)^{\frac{3}{2}}.\,V_2\,.\frac{2\gamma+2}{(2\gamma+3)\,!}\left[1-\frac{4}{3\pi}.\left(\frac{a}{c}\right)^3\right]$$

(*ii*) If the fifth power of $\frac{a}{c}$ is also retained, then,

$$B_{2\gamma+1}-(-)^{\gamma}\frac{2\gamma+2}{(2\gamma+3)\,!}\left(\frac{2a}{\pi}\right)^{\frac{3}{2}}V_2=(-)^{\gamma+1}.\frac{2}{\pi}\,\frac{1}{(2\gamma+1)\,!}$$

$$\left[\frac{1}{(2\gamma+3)}\left(\frac{2B_1}{k^3}+\frac{4!B_3}{k^5}\right)-\frac{4B_1}{(2\gamma+5)k^5}\right],$$

whence

$$B_1=\left(\frac{2a}{\pi'}\right)^{\frac{3}{2}}\frac{V_2}{3}\left[1-\frac{4}{3\pi k^3}+\frac{16}{5\pi k^5}\right]\quad k=\frac{a}{c}.$$

$$B_3 = -\left(\frac{2a}{\pi}\right)^{\frac{3}{2}} \frac{V_2}{30}\left[1 - \frac{4}{3\pi k^3} + \frac{368}{105\pi k^5}\right]$$

and $$B_{2\gamma+1} = (-)^\gamma \frac{2\gamma+2}{(2\gamma+3)!}\left(\frac{2a}{\pi}\right)^{\frac{3}{2}} V_2 \left[1 - \frac{4}{3\pi k^3} + \frac{16\,(8\gamma+15)}{15\pi k^5\,(2\gamma+5)}\right]$$

The Velocity-potential of the Motion.

The velocity-potential of the motion is, by (6)

$$\phi = \sum^{\infty}\left[b_{2n+1}\, P_{2n+1}^{(\mu)}\, q_{2n+1}^{(\zeta)} + a_{2n+1}\, P_{2n+1}^{(\mu_1)}\, q_{2n+1}^{(\zeta_1)} \right]$$

$$= \sqrt{\frac{\pi b}{2}} \int_0^\infty e^{-\lambda\zeta} J_0(\lambda\rho) \frac{d\lambda}{\sqrt{\lambda}} \sum_0^\infty b_{2n+1} J_{2n+\frac{3}{2}}^{(\lambda b)} + \sqrt{\frac{\pi a}{2}}$$

$$\int_0^\infty e^{-\lambda\zeta_1} J_0(\lambda\rho) \frac{d\lambda}{\sqrt{\lambda}} \sum_0^\infty a_{2n+1} J_{2n+\frac{3}{2}}^{(\lambda a)}.$$

$$= \sqrt{\frac{\lambda}{2}} \int_0^\infty J_0(\lambda\rho) \frac{d\lambda}{\sqrt{\lambda}}\left[\sqrt{\frac{a}{b}}\, f(\lambda a) e^{-\lambda\zeta_1} + \right.$$

$$\left. \sqrt{\frac{b}{a}}\, F(\lambda b)\, e^{-\lambda\zeta} \right]$$

$$a = b, \quad = \sqrt{\frac{\pi}{2}} \int_0^\infty J_0(\lambda\rho) \frac{d\lambda}{\sqrt{\lambda}} f(\lambda a)\left(e^{-\lambda\zeta_1} - e^{\lambda\zeta}\right), \qquad \text{from (B)}$$

where

$$f(z) = \sqrt{a} \sum^{\infty} a_{2n+1} J_{2n+\frac{3}{2}}^{(z)}$$ as has been found above;

and $$a_{2r+1} = -\int_0^\infty e^{-xc} \frac{dx}{x\sqrt{a}} f(xa)\,(4r+3)\, J_{2r+\frac{3}{2}}^{(xa)},$$

except when $r=0$, in which case, $\frac{2av_2}{\pi}$ must be added on the right-hand side.

Steady Rotation of Two Parallel Co-axial Circular Discs in a Viscous Liquid.

Introduction.

The rotation of a single circular disc has been considered by Dr. G. B. Jeffery.* We have contemplated the rotation of two circular discs, equal and unequal. In the case of unequal discs, a method of approximation has been followed, while in the case of equal circles the method of approximation as well that of integral equation of the previous problem have been introduced.

As before, the Fourier-Bessel Integral of spheroidal Harmonics has been necessary but Dr. Nicholson's † method of deriving the integral as applied to spheroidal Harmonics of zonal types seems to be unsuitable, when spheroidal Harmonics of associated types are concerned. By using some of his formulae, also otherwise, we have succeeded in having the corresponding Fourier-Bessel Integral. A general formula of which Bauer's ‡ formula is a special case has been obtained. The couple necessary to maintain the rotation has been ascertained.

* Proc. Lond. Math. Soc., 1915, p. 327.

† Phil. Trans. Royal Soc., Vol. 224, 1924, p. 320.

‡ 'Münchener Sitzungsber.' V. (1875), p. 263.

PART I.

Two Equal Circles.

We consider, as in the preceding problem, two equal spheroids (both oblate) rotating with equal angular velocity Ω about the common axis of revolution; (ζ, μ, ω) and (ζ_1, μ_1, ω) denote, as before, the oblate spheroidal co-ordinates of a point in space, (z, ρ, ω) and (z_1, ρ, ω) denote the corresponding cylindrical co-ordinates, with respect to the centres of the two spheroids as origin, the z-axes being measured positively in opposite senses along the axis of revolution; a, the radius of confocality, and c, the central distance. The spheroids being equal must have the same ζ, say ζ_0. Then, with the usual idea, we have the geometrical relations

$$z+z_1=c$$

$$i.e.,\ \mu\zeta+\mu_1\zeta_1=c/a, \text{ and } \frac{\rho}{a}=\sqrt{1-\mu^2}\ \sqrt{\zeta^2-1}$$

$$=\sqrt{1-\mu^2}\ \sqrt{\zeta_1^2+1}, \qquad \dots \quad (1)$$

Dr. G. B. Jeffery * has shown that, when squares and products of velocities can be neglected, v, the velocity in the direction, ω, (the other components$=0$), satisfies the equation

$$\Delta^2(\mathrm{v}\sin\omega)=0,$$

so that the detemination of v ensures the solution of the problem.

There being no slipping over the boundary, the boundary condition on either of the spheroids is

$$\mathrm{v}=a\Omega\sqrt{\zeta_0^2+1}\,\frac{\sqrt{1-\mu^2}}{\sqrt{1-\mu^2_1}}=a\Omega\sqrt{\zeta_0^2+1}\,\frac{\mathrm{P}^1{}_1(\mu)}{\mathrm{P}^1{}_1(\mu_1)}.$$

v, therefore, satisfies the following conditions:—

(*i*) $\Delta^2(\mathrm{v}\sin\omega)=0.$

(*ii*) $\mathrm{v}=0$, at infinity,

* Proc. Lond. Math. Soc., 1915, p. 327.

(*iii*) $v = a\Omega\sqrt{\zeta_0^2+1}\,P'_1(\mu)$, on one spheroid,

(*iv*) $v = a\Omega\sqrt{\zeta_0^2+1}\,P'_1(\mu_1)$, on the other.

Fourier-Bessel Integral of a Spheroidal Harmonic of Associated Type.

Deriving the analogue of the well-known formula*

$$P^m{}_n(\mu)Q^m{}_n(\zeta) = \frac{1}{\pi}\,\frac{(n+m)!}{(n-m)!}$$

$$\int_0^\pi Q_n[\mu\zeta + \sqrt{1-\mu^2}\,\sqrt{1-\zeta^2}\cos u]\cos mu\,du,$$

in q-functions, we have

$$P_n{}^m(\mu)q_n{}^m(\zeta) = \frac{i^m}{\pi}\,\frac{(n+m)!}{(n-m)!}$$

$$\int_0^\pi q_n[\mu\zeta - i\sqrt{1-\mu^2}\,\sqrt{\zeta^2-1}\,\cos u]\cos mu\,du.$$

$$= \frac{i^m}{\pi}\,\frac{(n+m)!}{(n-m)!}$$

$$\int_0^\pi q_n\left[\frac{z-i\rho\cos u}{a}\right]\cos mu\,du,\ \text{by} \qquad \dots \quad (1),$$

$$= \frac{i^m}{\pi}\,\frac{(n+m)!}{(n-m)!}\sqrt{\frac{\pi a}{2}}\int_0^\pi$$

$$\int_0^\infty e^{-\lambda(z-i\rho\cos u)}\,.\,J_{n+\frac{1}{2}}^{(\lambda a)}\,\frac{d\lambda}{\sqrt{\lambda}}\,.\,\cos mu\,du\ \ddagger$$

by a change of order,

$$= \frac{i^m}{\pi}\,\frac{(n+m)!}{(n-m)!}\sqrt{\frac{\pi a}{2}}\int_0^\infty e^{-\lambda z}\,J_{n+\frac{1}{2}}^{(\lambda a)}\,\frac{d\lambda}{\sqrt{\lambda}}$$

* Proc. Edin. Math. Soc., Vol. 33, p. 119, Dr. G. B. Jeffery.

† ,, ,, ,, ,, ,, p. 68. Blades.

‡ *Loc. cit.* Dr. Nicholson's Memoir,

$$\int_0^\pi \cos mudu \left[J_0^{(\lambda\rho)} + 2 \sum_{s=1}^{\infty} i\, J_s^{(\lambda\rho)} \cos su \right]$$

$$= (-)^m \frac{(n+m)!}{(n-m)!} \sqrt{\frac{\pi a}{2}}$$

$$\int^\infty e^{-\lambda z}\, J_{n+\frac{1}{2}}^{(\lambda a)}\, J_m^{(\lambda\rho)} \frac{d\lambda}{\sqrt{\lambda}} \qquad \ldots \quad (2)$$

Transformation of a Spheroidal Harmonic of Associated Type corresponding to a Change of Origin.

$$P^m{}_n(\mu_1) q^m{}_n(\zeta_1) = \frac{i^m}{\pi} \frac{(n+m)!}{(n-m)!}$$

$$\int_0^\pi q_n[\mu_1\zeta_1 - i\sqrt{1-\mu^2{}_1}\sqrt{\zeta^2{}_1+1}\cos u] \cos mudu.$$

by (1),

$$= f(m, n) \int_0^\pi q_n \left[\frac{c}{a} - \mu\zeta - i\sqrt{1-\mu^2}\sqrt{\zeta^2+1}\cos u \right] \cos mudu \;;$$

where

$$f(m, n) = \frac{i^m}{\pi} \cdot \frac{(n+m)!}{(n-m)!}$$

$$= f(m, n) \sum_{r=0}^{\infty} (-)^r 2^n \frac{(n+r)!(n+2r)!}{r!(2n+2r+1)!}$$

$$\int_0^\pi \frac{\cos mudu}{\left[\frac{c}{a} - \mu\zeta - i\sqrt{1-\mu^2}\sqrt{\zeta^2+1}\cos u \right]^{n+2r+1}}$$

$$D = a\frac{\partial}{\partial c}, \quad = (fm, n) \sum_{r=0}^{\infty} 2^n (-)^{n+r}_{\zeta} \frac{(n+r)!D^{n+2r}}{r!(2n+2r+1)!}$$

* *Loc. cit.* Theory of Bessel-Function, Watson, p. 22.
,, ,, Hydrodynamics, Lamb, p. 478.

$$\int_0^{\pi} \frac{\cos mudu}{\frac{c}{a} - \mu\zeta - i\sqrt{1-\mu^2}\ \sqrt{\zeta^2+1}\ \cos u}$$

$$c>a, = \mathrm{f}(m, n)(-)^n \sqrt{\frac{\pi}{2\mathrm{D}}}\, \mathrm{J}_{n+\frac{1}{2}}^{(\mathrm{D})} \,.\, i$$

$$\int_0^{\pi} \sum_{r=0}^{\infty} (2r+1)\mathrm{Q}_r\left(\frac{ci}{a}\right)\mathrm{P}_r[i\mu\zeta - \sqrt{1-\mu^2}\ \sqrt{\zeta^2+1}\ \cos u]\cos mudu.$$

$$=(-)^n \frac{(n+m)!}{(n-m)!} \sqrt{\frac{\pi}{2\mathrm{D}}}\, \mathrm{J}_{n+\frac{1}{2}}^{(\mathrm{D})}$$

$$\sum_{r=0}^{\infty} (-)^m(2r+1)\frac{(r-m)!}{(r+m)!}\, q_r\left(\frac{c}{a}\right)\mathrm{P}_r^m(\mu)_r p_r^m(\zeta)^+$$

$$= \sum_{r=m}^{\infty} (2r+1)\omega^m{}_n(r, c, a)\mathrm{P}_r{}^m(\mu)p_r{}^m(\zeta), \qquad \ldots \quad (3)$$

where

$$\omega^m{}_n(r, c, a) = (-)^{m+n}\frac{(r-m)!}{(r+m)!}\ \frac{(n+m)!}{(n-m)!} \sqrt{\frac{\pi}{2\mathrm{D}}}\, J_{n+\frac{1}{2}}^{(\mathrm{D})}\, q_r\left(\frac{c}{a}\right)$$

introducing the integral expression for q_r,

$$=(-)^{m+n}\frac{(r-m)!}{(r+m)!}\ \frac{(n+m)!}{(n-m)!}\ \frac{\pi}{2} \sqrt{\frac{a}{\mathrm{D}}}\, \mathrm{J}_{r+\frac{1}{2}}^{(\mathrm{D})}$$

$$\int_0^{\infty} e^{-\lambda c}\, \mathrm{J}_{r+\frac{1}{2}}^{(\lambda a)}\, \frac{d\lambda}{\sqrt{\lambda}}$$

$$=(-)^m \frac{(r-m)!}{(r+m)!}\ \frac{(n+m)!}{(n-m)!}\ \frac{\pi}{2}$$

$$\int_0^{\infty} e^{-\lambda c}\, \mathrm{J}_{r+\frac{1}{2}}^{(\lambda a)}\, \mathrm{J}_{n+\frac{1}{2}}^{(\lambda a)}\, \frac{d\lambda}{\lambda} \qquad \ldots \quad (4)$$

Formation of an integral equation.

Assume

$$V=\sum_{n=1}^{a} a_n\left[\, p^1{}_n(\mu)q^1{}_n(\zeta)+p^1{}_n(\mu_1)q^1{}_n(\zeta_1)\,\right], \qquad \ldots \quad (5)$$

which evidently satisfies the conditions (i) and (ii).

Near the surface of the spheroid (center, at $z=0$), we have by (3),

$$V=\sum_{r=1}^{\infty} a_r p^1{}_r(\mu)q^1{}_r(\zeta)+\sum_{n=1}^{\infty} a_n \sum_{r=1}^{a}(2r+1)\;\omega^1{}_n\;(r,\,c,\,a)\;P^1{}_r(\mu)p^1{}_r(\zeta_0).$$

The boundary condition over it gives,

$$a\sqrt{\zeta_0{}^2+1}\;\;P'_1(\mu)\Omega=\sum_{r=1}^{\infty} a_r P^1{}_r(\mu)P^1{}_r(\zeta_0)$$
$$+\sum_{n=1}^{\infty} a_n \sum_{r=1}^{\infty}(2r+1)\;\omega^1{}_n\;(r,\,c,\,a)\;P^1{}_r(\mu)p^1{}_r(\zeta),$$

whence, equating the co-efficients of the associated functions,

$$\left.\begin{aligned} a\Omega\sqrt{\zeta_0{}^2+1}\;&=a_1 q^1{}_1(\zeta_0)+3p^1{}_1(\zeta_0)\sum_{n=1}^{a} a_n\;\omega^1{}_n\;(r,\,c,\,a),\\ 0\;&=a_r q^1{}_r(\zeta_0)+(2r+1)\;p^1{}_r\;(\zeta_0)\;\sum_{n=1}^{a} a_n\omega^1{}_n\;(r,\,c,\,a),\;\; r>1 \end{aligned}\right\}\ldots(x)$$

Therefore,

$$a_r=-(2r+1)\frac{p^1{}_r(\zeta_0)}{q^1{}_r(\zeta_0)}\;\sum_{n=1}^{a} a_n\omega^1{}_n\;(r,\,c,\,a);\;\; r=1,\,2,\,3,\,4,\ldots ad\;inf.$$

when $r=1$, there must be an addition of $\dfrac{a\Omega\sqrt{\zeta_0{}^2+1}}{q^1{}_1(\zeta_0)}$, on the right-hand side.

Two circular discs.

In the case of two circular discs, we have $\zeta_0=0$. Again, as before,* $\dfrac{p'_r(0)}{q'_r(0)}=0,\; -\dfrac{2}{\pi}$, according as r is even or odd respectively;

* *Loc. cit.*

and $q'_1(0) = -\frac{\pi}{2}$. Consequently we notice that the co-efficients a_{2r} do not occur in v, and putting $r=2m+1$, we have,

$$a_{2m+1} = \frac{2}{\pi}(4m+3)\sum_{n=0}^{a} a_{2n+1}\,\omega'_n\,(2m+1,\,c,\,a)$$

$$= -\frac{4m+3}{(2m+1)(2m+2)}\int_0^\infty J_{2m+3/2}^{(\lambda a)} \sum_{n=0}^{a} J_{2n+3/2}^{(\lambda a)}\,.$$

$$e^{-\lambda c}.\frac{d\lambda}{\lambda}.\ (2n+1)(2n+2)a_{2n+1},\text{ by (4)}$$

$$= -\frac{4m+3}{(2m+1)2m+2)}\int_0^\infty e^{-\lambda c}\,J_{2m+3/2}^{(\lambda a)}\,\frac{d\lambda}{\lambda}$$

$$\left\{\sum_{n=0}^{\infty}(2n+1)(2n+2)a_{2n+1}\,J_{2n+3/2}^{(\lambda a)}\right\}$$

$$= -\frac{4m+3}{(2m+1)(2m+2)}\int_0^\infty e^{-\frac{\lambda c}{a}}\,J_{2m+3/2}^{(\lambda a)}\,F(\lambda)\frac{d\lambda}{\lambda}, \qquad \ldots (6)$$

where $$F(x) = \sum_{n=0}^{a}(2n+1)(2n+2)\,a_{2n+1}\,J_{2n+3/2}^{(x)} \qquad \ldots (7)$$

Introducing, as before, a new variable, y, independent of x, changing λ into x, multiplying by$(2m+1)(2m+2)\ J_{2m+3/2}^{(y)}$, and summing for all integral values of m, we have,

$$F(y) + \frac{4a\Omega}{\pi}\,J_{3/2}^{(y)} = -\int_0^\infty e^{-\frac{xc}{a}}\,F(x)\frac{dx}{x}\sum_{m=0}^{\infty}(4m+3)\,J_{2m+3/2}^{(x)}\,J_{2m+3/2}^{(y)}.$$

$$= -\sqrt{\frac{1}{\pi}xy}\int_0^\infty e^{-\frac{xc}{a}}\,F(x)\frac{dx}{x}\left[\frac{\sin(x-y)}{x-y} - \frac{\sin(x+y)}{x+y}\right]$$

$$\ldots (8)$$

The solution.

This is the same integral equation as that previously discussed and requires no further analysis. We, however, state the results that are obtained. Assuming $\frac{f(y)}{\sqrt{y}} = \sum_{0}^{a} b_n y^n$, it might be seen, as before, that the co-efficients b_{2r} do not occur and

(*i*) retaining only the third power of a/c, we may have,

$$b_1 = -\frac{2a\Omega}{3} \cdot \left(\frac{2}{\pi}\right)^{\frac{3}{2}} \left[1 - \frac{4}{8\pi}\left(\frac{a}{c}\right)^3\right].$$

and $$b_{2r+1} = (-)^{r+1} \cdot \frac{2r+2}{(2r+3)!} \cdot \frac{4a\Omega}{\pi}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left[1 - \frac{4}{3\pi}\left(\frac{a}{c}\right)^3\right]$$

(*ii*) and retaining the fifth-power of a/c also, we may have,

$$b_1 = -\frac{2a\Omega}{3} \cdot \left(\frac{2}{\pi}\right)^{\frac{3}{2}} \cdot \left[1 - \frac{4}{3\pi k^3} + \frac{16}{5\pi k^5}\right], \quad \kappa = c/a .$$

$$b_3 = \frac{2a\Omega}{30} \cdot \left(\frac{2}{\pi}\right)^{\frac{3}{2}}\left[1 - \frac{4}{3\pi k^3} + \frac{368}{105\pi k^5}\right],$$

and $$b_{2r+1} = (-)^{r+1} \cdot 2a\Omega \cdot \left(\frac{2}{\pi}\right)^{\frac{3}{2}} \cdot \frac{2r+2}{(2r+3)!}\left[1 - \frac{4}{3\pi k^3} + \frac{16(8r+15)}{15(2r+5)\pi k^5}\right].$$

The value of v.

We have, from (5) and (2),

$$V = \sum^{a} a_{2n+1}\left[p^1_{2n+1}(\mu)\, q^1_{2n+1}(\zeta) + p^1_{2n+1}(\mu_1)\, q^1_{2n+1}(\zeta_1)\right]$$

$$= -\sqrt{\frac{\pi a}{2}} \int_0^\infty \left(e^{-\lambda z_1} + e^{-\lambda z}\right) J_1(\lambda\rho)\, \frac{d\lambda}{\sqrt{\lambda}} \sum^{\infty} (2n+1)(2n+2) a_{2n+1} J_{2n+\frac{3}{2}}(\lambda a)$$

$$= -\sqrt{\frac{\pi a}{2}} \int_0^\infty \left(e^{-\lambda z_1} + e^{-\lambda z}\right) J_1(\lambda\rho)\, f(\lambda a)\, \frac{d\lambda}{\sqrt{\lambda}}$$

where

$$f(x)=\sum_{}^{\infty}(2n+1)(2n+2)\,a_{2n+1}J^{(x)}_{2n+\frac{3}{2}},$$ has been found above;

and $$a_{2m+1}=-\frac{4m+3}{(2m+1)(2m+2)}\int_{o}^{\infty}e^{-\frac{\lambda c}{a}}J^{(\lambda)}_{2m+\frac{3}{2}}f(\lambda)\frac{d\lambda}{\lambda},$$

except when $m=o$, in which case, $-\frac{2a\Omega}{\pi}$ must be added on the right hand side.

Alternative method of determining the co-efficients.

The equations (x) may be solved differently thus:—

as previously observed, the co-efficients a_{2r} vanishing, we have to solve,

$$\left.\begin{aligned}\frac{2a\Omega}{\pi}&=a_1-\frac{6}{\pi}\sum_{o}^{\infty}a_{2n+1}\,\omega^1_{2n+1}(1,c,a),\\ 0&=a_{2r+1}-\frac{2}{\pi}(4r+3)\sum_{o}^{\infty}a_{2n+1}\,\omega^1_{2n+1}(2r+1,c,a),\ r>0\end{aligned}\right\}$$

we may obtain from (4),

$$\omega^1_{2n+1}(2r+1,c,a)=-2^{2r+2n+2}\cdot\frac{(2n+1)(2n+2)(2n+1)!(2r+1)!}{(2r+1)(2r+2)(4n+3)!(4r+3)!}$$

$$\left[(2r+2n+2)!\left(\frac{a}{c}\right)^{2r+2n+3}-(2r+2n+4)!\left(\frac{a}{c}\right)^{2r+2n+5}\left\{\frac{1}{2(4r+5)}+\frac{1}{2(4n+5)}\right\}+\ldots\right]\quad\ldots\quad(9)$$

whence *

(i) neglecting terms of order $\left(\frac{\text{Linear dimensions}}{\text{Central distance}}\right)^3$ and higher,

$$a_1=-\frac{2a\Omega}{\pi};\ a_{2r+1}=0,\ r>0.$$

* The co-efficients are supposed to be expanded in positive integral powers of a/c on physical grounds.

The corresponding value of V is

$$V=-\frac{2a\Omega}{\pi}[p^{1}_{1}(\mu)\ q^{1}_{1}(\zeta)+p^{1}_{1}(\mu)\ q^{1}_{1}(\zeta_1)]$$

(*ii*) neglecting terms of order $\left(\frac{\text{Linear dimensions}}{\text{Central distance}}\right)^5$ and higher

$$a_1=-\frac{2a\Omega}{\pi}\left[1-\frac{4}{\pi}\left(\frac{a}{c}\right)^3\right]\ ;\ a_{2r+1}=0\ ;\ r>0$$

(*iii*) neglecting terms of order $\left(\frac{\text{Linear dimensions}}{\text{Central distance}}\right)^7$ and higher

$$a_1=-\frac{2a\Omega}{-}\left[1+\frac{16}{a\pi^2}\ \frac{a^3b^3}{c^6}-\frac{24}{\pi}\left\{\frac{1}{3}\left(\frac{a}{c}\right)^3+\frac{18}{5}\left(\frac{a}{c}\right)^5\right\}\right]$$

$$a_3=\frac{16a\Omega}{45\pi^2}\left(\frac{a}{c}\right)^5$$

$$a_{2r+1}=0.\ r\geqq 2.$$

Part II

Two unequal circles.

With the preliminary statements made in Part I, the polar axes being measured in opposite senses along the common axis of revolution, we may, with the aid of the geometrical relations

$$a\mu\zeta+b\mu_1\zeta_1=c \text{ and } a\sqrt{1-\mu^2}\ \sqrt{\zeta^2+1}=b\sqrt{1-\mu_1^2}\ \sqrt{\zeta_1^2+1},$$

deduce the following transformations:—

$$P^m_n(\mu_1)q^m_n(\zeta_1)=\sum_{r=m}^{\infty}(2r+1)\ {}_1\omega^m_n(r.\ c.\ a,\ b)\ P^m_n(\mu)p^m_n(\zeta) \quad \ldots \quad (11)$$

where

$${}_1\omega^m_n(r,\ c.\ a,\ b)=(-)^{n-m}\frac{b}{a}\ \frac{(r-m)!}{(r+m)!}\ \frac{(n+m)!}{(n-m)!}$$

$$\sqrt{\frac{\pi}{2D}}J^{(D)}_{n+\frac{1}{2}}\ q_r\left(\frac{c}{a}\right),\ D=b\frac{\partial}{\partial c} \quad \ldots \quad (12)$$

and

$$P_n^m(\mu)q_n^m(\zeta)=\sum (2r+1)\ {}_2\omega_n^m(r,\ c,\ b,\ a)\ P_n^m(\mu_1)p_n^m\ (\zeta_1) \qquad (13)$$

where

$${}_2\omega_n^m(r.\ c,\ b,\ a)=(-)^{n+m}\frac{a}{b}\ \frac{(r-m)\,!}{(r+m)\,!}\ \frac{(n+m)\,!}{(n-m)\,!}$$

$$\sqrt{\frac{\pi}{2D'}}\ J_{n+\frac{1}{2}}^{(D')}\ q_r\left(\frac{c}{b}\right)\ D'=a\frac{\partial}{\partial} \qquad (14)$$

Seeking according to the principle of Dr. G. B. Jeffery, a solution of the problem, we assume,

$$V=\sum_{n=1} \left[A_r P^1_n(\mu)q^1_n(\zeta)+B_n P^1_n(\mu_1)q^1_n(\zeta_1)\right] \qquad \ldots\ (15)$$

Near the surface of the spheroid, a, ($\zeta=\zeta_\circ$), we have by (11),

$$V=\sum_{r=1}^{\infty} ArP^1_r(\mu)q^1r(\zeta)+\sum_{n=1}^{\infty} B_n \sum_{r=1}^{\infty} (2r+1)\ {}_1\omega^1_n(r.c.a,b)P^1_r(\mu)p^1_r(\zeta)$$

whence the boundary condition over it gives, by equating the co-efficients of the Harmonics,

$$a\Omega\sqrt{\zeta_0^2+1}\ =A_1q^1_1(\zeta_0)+3p^1_1(\zeta_0)\ \sum B_n\ {}_1\omega^1_n(1.\ c,a,b) \qquad \ldots\ A$$

$$O=A_r\ q^1_r(\zeta_0)+(2r+1)p^1_r(\zeta_0)\sum_{n=1} B_n\ {}_1\omega^1_n(r.\ c,\ a,b),\ r>1$$

The boundary condition over the other spheroid $\zeta_1={}_1\zeta_0$, gives, by (13)

$$b\Omega_1\ \sqrt{{}_1\zeta_0^2+1}=B_1q^1_1(\xi_0)+3\sum_{n=1} A_n\ {}_2\omega^1_n\ (1,\ c.\ b,\ a)\ p^1_1(\zeta_0),$$

$$O=B_rq^1_r({}_1\zeta_0)+(2r+1)p^1_r({}_1\zeta_0)\sum_{n=1} A_n\ {}_2\omega^1_n(r.\ c,\ b.\ a),\ r>1 \qquad \ldots\ B$$

The boundary condition on the spheroids, subject to no slipping, being

$V =$ velocity in the direction of $\omega = a\Omega\sqrt{1-\mu^2}\sqrt{\zeta_0^2+1}$

$= a\Omega\sqrt{\zeta_0^2+1}\ P^1_1(\mu)$ over the spheroid, $\zeta=\zeta$.

$V = b\Omega_1\sqrt{{}_1\zeta_0^2+1}\ P^1_1(\mu_1)$, over the spheroid $\zeta = {}_1\zeta_0$,

Ω, Ω_1, denoting their angular velocities.

Two unequal parallel circular discs.

If $\zeta_0 = {}_1\zeta_0 = 0$, the two spheroids reduce to two un-equal circular discs, of radii a and b respectively; since $\dfrac{p'_{2n}(o)}{q'_{2n}(o)} = 0$,

$p'^{(o)}_{2n+1}/q'^{(o)}_{2n+1} = -\dfrac{2}{\pi}$; $1/q'(o) = -2/\pi$. the equation A and B become

$$\left.-\frac{2a\Omega}{\pi} = A_1 - \frac{6}{\pi}\sum_{n=o}^{\infty} B_{2n+1}\ {}_1\omega^1_{2n+1}\ (1,\ c,\ a.\ b),\ \right\} \qquad \ldots\ A'$$

$$0 = A_{2r+1} - \frac{2}{\pi}\ (4r+3)\sum_{n=o}^{\infty} B_{2n+1}\ {}_1\omega^1{}_{2n+1}\ (2r+1,\ c,\ a,\ b), r70$$

$$0 = A_{2r}$$

and $$-\frac{2b\Omega_1}{\pi} = B_1 - \frac{6}{\pi}\sum_{n=o}^{\infty} A_{2n+1}\ {}_2\omega^1{}_{2n+1}\ (1_{\varepsilon}\ c.b.a).$$

$\ldots\ B'$

$$0 = B_{2r+1} - \frac{2}{\pi}\ (4r+3)\sum_{n+o}^{\infty} A_{2n+1}\ {}_2\omega^1{}_{2n+1}\ (2r+1,\ c,\ a,\ b),\ r70$$

$$0 = A_{2r}$$

whence, from A′ and B′, by substitution,

$$A_1 - \sum_{n=o}^{\infty} A_{2n+1}\ \theta\ {}^{2n+1}_{1} = -\frac{2a\Omega}{\pi} - \frac{12b\Omega_1}{\pi^2}\ {}_1\omega^1{}_1\ (1,\ c,\ a,\ b),$$

$$A_{2r+1} - \sum_{n=o}^{\infty} A_{2n+1}\ \theta 2_n + 1, {}_{2r+1} = -\frac{4b\Omega_1}{\pi 2}\ (4r+3)\ {}_1\omega^1{}_1(2r+1,\ c,a,b)$$

$r70$

where

$$\theta\underset{2r+1}{2n+1,} = \frac{4}{\pi 2}(4r+3)\sum_{p=o}^{\infty}(4p+3)\ {}_1\omega^1{}_{2p+1}(2r+1,\ c,a,b)\ {}_2\omega^1{}_{2n+1}(2p+1,\ c,b,a).$$

And

$$B_1 - \sum_{n=o}^{\infty} B_{2n+1}\ \phi\underset{1}{2n+1,} = -\frac{2b\Omega_1}{\pi} - \frac{12a\Omega}{\pi 2}\ {}_2\omega^1{}_1\ (1,\ c,b.a),$$

$$B_{2r+1} - \sum_{n=o}^{\infty} B_{2n+1}\ \phi\underset{2r+1}{2n+1,} = -\frac{4a\Omega_1}{\pi 2}(4r+3)\ {}_2\omega^1{}_1(2r+1,\ c,b,a),$$

$r70$

where

$$\phi\underset{2r+1}{2n+1,} = \frac{4}{\pi 2}(4r+3)\sum_{p=o}^{\infty}(4p+3)\ {}_1\omega^1{}_{2n+1}(2p+1,\ c\ a,b)\ {}_2\omega^1{}_{2p+1}(2r+1,\ c,b,a).$$

Approximate determinations of the co-efficients.

We may have, from (12),

$${}_1\omega^1{}_{2n+1}(2r+1,\ c,a,b) = -2^{2r+2n+2}\frac{(2n+1)(2n+2)(2n+1)\,!(2r+1)\,!}{(2r+1)(2r+2)(4r+3)\,!(4n+3)\,!}$$

$$(2r+2n+2)\,!\ \frac{b^{2n+2}\ a^{2r+1}}{c^{2r+2n+3}} - (2r+2n+4)\,!\ \frac{b^{2n+2}\ a^{2r+1}}{c^{2r+2n+5}}$$

$$\left\{\frac{b^2}{2(4n+5)} + \frac{a^2}{2(4r+5)} + \dots\right\} + \quad \Big]$$

and from (14),

${}_2\omega^1{}_{2n+1}\ (2r+1,\ c,\ b,\ a)=$ a similar expression as above in which a and b are interchanged.

$$\theta\underset{2r+1}{2n+1,} = \frac{4}{\pi 2}(4r+3)\sum_{p=o}^{\infty}(4p+3).2^{2r+2n+4p+4}$$

$$\frac{(2p+2n+2)\,!\ (2p+2r+2)\,!\ (2n+1)(2n+2)(2n+1)\,!\ (2r+1)\,!(2p+1)\,!^2}{(2r+1)(2r+2)(4r+3)\,!\ (4n+3)\,!\ \ (4p+3)\,!^2}$$

$$\frac{a^{2n+2r+3}b^{4p+3}}{c^{2n+2r+4p+6}} +$$

$\phi 2_{n}+1,_{2r+1}$ = a similar expression in which a, b, are interchanged +.

$$\theta_{1,1}=\frac{16}{9\pi^2}\frac{a^3b^3}{c^6}+\ldots\ldots\ldots;\quad \phi_{1,1}=\frac{16}{9\pi^2}\frac{a^3b^3}{c^6}+\ldots\ldots\ldots$$

whence *

(i) neglecting terms of order $\left(\dfrac{\text{Linear dimensions}}{\text{Central distance}}\right)^3$ and higher,

$$A_1=-\frac{2a\Omega}{\pi}\quad;\quad A_{2r+1}=0,\ r>0;$$

$$B_1=-\frac{2b\Omega_1}{\pi};\quad B_{2r+1}=0,\ r>0.$$

The corresponding value of ν is from (15),

$$\nu=-\frac{2}{\pi}\left[a\Omega P^1{}_1(\mu)q^1{}_1(\zeta)+b\Omega_1P^1{}_1(\mu_1)q^1{}_1(\zeta_1)\right]$$

(ii) neglecting terms of order $\left(\dfrac{\text{Linear dimensions}}{\text{Central distance}}\right)^5$ and higher,

$$A_1=-\frac{2a}{\pi}\left[\Omega-\frac{4\Omega_1}{3\pi}\frac{b^3}{c^3}\right];\ A_{2r+1}=0,\ r>0\ ;$$

$$B_1=-\frac{2b}{\pi}\left[\Omega_1-\frac{4\Omega}{3\pi}\frac{a^3}{c^3}\right];\ B_{2r+1}=0,\ r>0\ ;$$

(iii) neglecting terms of order $\left(\dfrac{\text{Linear dimensions}}{\text{Central distance}}\right)^7$ and higher,

$$A_1=-\frac{2a\Omega}{\pi}\left[1+\frac{16}{9\pi^2}\frac{a^3b^3}{c^6}\right]+\frac{8a\Omega_1}{3\pi^2}\frac{b^3}{c^3}\left(1-\frac{6}{5}\frac{a^2+b^2}{c^2}\right);$$

$$A_3=\frac{16b\Omega_1}{45\pi^2}\frac{a^3b^2}{c^5};$$

$$A_{2r+1}=0,\ r\geq 2\ ;$$

* The co-efficient A's and B's are supposed to be expanded in positive integral powers of $\dfrac{\text{Linear dimension}}{\text{Central distance}}$ on physical grounds.

$$B_1 = -\frac{2b\Omega_1}{\pi}\left[1+\frac{16}{9\pi^2}\frac{a^3b^3}{c^6}\right]+\frac{8b\Omega}{3\pi^2}\frac{a^3}{c^3}\left(1-\frac{6}{5}\frac{a^2+b^2}{c^2}\right);$$

$$B_3 = \frac{16a\Omega}{45\pi^2}\frac{a^2b^3}{c^5}$$

$$B_{2r+1}=0,\ r>2.$$

The corresponding value of v can be written down at once.

The couple necessary to maintain the relation.

If $(\xi.\ \eta,\ \omega)$ be a system of orthogonal co-ordinates, the elements of arcs measured along the normals to the surfaces $\xi.\ \eta,\ \omega=$const. are $\frac{d\xi}{h_1}.\ \frac{d\eta}{h_2},\ \frac{d\omega}{h_3}$, respectively ; and if $(u',\ v',\ \omega')$ denote the corresponding components of velocity, we have, with the usual notation of stress-components, $\widehat{\xi\omega}=\mu.\left[\frac{h_3}{h_1}\frac{\partial}{\partial\omega}(h_1u')+\frac{h_1}{h_3}\frac{\partial}{\partial\xi}(h_3\omega')\right]$, where $\mu_o=$co-efficient of viscocity. Moreover, if $(\xi,\ \eta)$ be conjugate functions of $(z,\ \rho)$, we have, $h_1=h_2$ and $h_3=\frac{1}{\rho}$ and if the solid be defined by $\xi=$const. we have, for the tangential stress on its surface in the direction perpendicular to the axis $\mu.h_1\rho\left(\frac{\partial}{\partial\xi}\frac{V}{\rho}\right)$ and the total couple exerted by the fluid on the solid is

$$G_1=2\pi\mu.\int\rho^3\frac{\partial}{\partial\xi}\left(\frac{V}{\rho}\right)d\eta$$

the value of the integrand being taken on the surface of the solid and the integration extending round the contour of the solid

$$=2\pi\mu.\int\rho^3\frac{\partial}{\partial\zeta}\left(\frac{V}{\rho}\right)_{\bullet}\frac{\partial\zeta}{\partial\xi}\frac{\partial\eta}{\partial\mu}d\mu, \qquad (y)$$

where $(\zeta,\ \mu,\ \omega)$ form an orthogonal system such that

$$\zeta=F_1(\xi)\text{ and }\mu=F_2(\eta).$$

In the case of an oblate spheroid we may take $\zeta = \sinh \xi$ and $\mu = \cos \eta$, so that the formula (y) takes the form

$$G_1 = -2\pi\mu. \int \rho^3 \frac{\partial}{\partial \xi}\left(\frac{V}{\rho}\right) \sqrt{\frac{\zeta^2+1}{1-\mu^1}}\; d\mu.$$

Thus, taking V say from (*iii*) of the preceding section, we have,

$$\frac{\partial}{\partial \zeta}\left(\frac{V}{\rho}\right) = \frac{A_1}{a} \frac{\partial^2 q_1}{\partial \zeta^2} + \frac{A_3}{a}, \frac{d^2 q_3}{d\xi^2}. \frac{P^1{}_3(\mu)}{\sqrt{1-\mu^2}}$$

$$+ \sum_0^\infty \frac{2r+1}{a} \frac{P^1{}_r(\mu)}{\sqrt{1-\mu^2}} \frac{d}{d\zeta}\left(\frac{P^1{}_r(\zeta)}{\sqrt{\zeta^2+1}}\right) [B_1\, {}_1\omega^1{}_1(r, c, a, b) +$$

$$B_3, \omega^1{}_3(r, c, a, b)]$$

and the couple on the circle a

$$G_1 = -2\pi\mu.\, a^2 A_1 \left(\frac{d^2 q_1}{d\zeta^2}\right), \int^1 (1-\mu^2)\, d\mu,$$

[the other terms vanishing, due to the well-known integral properties of the products of associated functions and also due to the fact that

$$\left\{\frac{\partial}{\partial \zeta}\left(\frac{P^1{}_r(\zeta)}{\sqrt{\zeta^2+1}}\right)\right\}_0 = 0. \quad]$$

$$= -\frac{16\pi\mu.}{3} a^2 A,$$

$$= \frac{32\mu.\, a^3}{3}\left[\left(\Omega(1 + \frac{16}{9\pi^2} \frac{a^3 b^3}{c^6}\right) - \frac{4\Omega_1}{3\pi} \frac{b^3}{c^3}\left(1 - \frac{6}{5} \frac{a^2+b^2}{c^2}\right)\right]$$

Observation.

(a) If terms of order $\left(\dfrac{\text{Linear dimension}}{\text{Central distance}}\right)^3$ and higher be neglected, we have from above

$G_1 = \dfrac{32\mu.\, a^3}{3}\, \Omega$, which is Dr. Jeffery's result for the rotation of a single circular disc. We observe that the couple on either of the discs is quite independent of the rotation of the other.

(*b*) If the circle b be constrained to rotate with a given spin Ω_1, the steady angular velocity with which the circular disc a would rotate, if allowed to move freely, would be that for which the couple on it vanishes. Thus, corresponding to (*iii*) of the preceding section, the steady angular velocity

$$\frac{\frac{4\Omega_1}{3\pi}\frac{b^3}{c^3}\left(1-\frac{6}{5}\frac{a^2+b^2}{c^2}\right)}{1+\frac{16}{9\pi^2}\frac{a^3b^3}{c^6}}$$

$$=\frac{4\Omega_1}{3\pi}\frac{b^3}{{}_3}\left[1-\frac{6}{5}\frac{a^2+b^2}{c^2}\right],\text{ to our order of approximate.}$$

A general formula.—(Further application of the transformation formula.)

Obtaining the transformation of spheroidal Harmonic in two different ways, an interesting general formula can be deduced, which includes Bauer's * formula as a special case, But it is derived from the equality of two integrals, which may not satisfy, in point of rigour, a fastidious modern mathematician.

Assuming, for simplicity, the radii of confocality to be equal to unity and the geometrical relations such as those previously stated, we have,

$$P_n^m(\mu)\,Q_n^m(\zeta)=\frac{1}{\pi}\frac{(n+m)!}{(n-m)!}\int_0^\pi Q_n[\mu\zeta+\sqrt{1-\mu^2}\,\sqrt{1-\zeta^2}\cos u]\cos mu\,du$$

$$=\frac{1}{\pi}\frac{(n+m)!}{(n-m)!}\sum_0^\infty 2^n\frac{(n+r)!(n+2r)!}{r!(2n+2r+1)!}\int^\pi$$

$$\frac{\cos mu\,du}{[c-\mu_1\zeta_1+\sqrt{1-\mu_1^2}\sqrt{1-\zeta_1^2}\cos u]^{n+2r+1}}$$

$D=\dfrac{\partial}{\partial c}$ multiplying and dividing by $-i$, $=\dfrac{1}{\pi}\dfrac{(n+m)!}{(n-m)!}\displaystyle\sum_0^\infty(-2)^n$

$$\frac{(n+r)!D^{n+2r}}{r!(2n+2r+1)!}(\mathbf{\ast}i\int_0^\pi\frac{\cos mu\,du}{-ic+i\mu_1\zeta_1-i\sqrt{1-\mu_1^2}\sqrt{1-\zeta_1^2}\cos u}$$

* Münchener sitzungsber, V (1875), p. 263.

since $\frac{1}{k}=\int_0^\infty e^{-kx}\,dx$, $=\frac{(-)^{n+1}}{\pi}\frac{(n+m)!}{(n-m)!}\sqrt{\frac{\pi}{2D}}\,I^{(D)}_{n+\frac{1}{2}}.i.\int_0^\infty d\lambda$

$$\int_0^\pi e^{i\lambda c-i\lambda\mu_1\zeta_1+i\lambda\sqrt{1-\mu_1^2}\sqrt{1-\zeta_1^2}\cos u}\cos mn\,du, \text{ on changing order,}$$

[using the well-known formula, $=(-)^{n+1}\frac{(n+m)!}{(n-m)!}\sqrt{\frac{\pi}{2D}}$

$$e^{i\lambda\rho\cos u}=J_0(\lambda\rho)+2\sum_{s=1} i^s J_s(\lambda\rho)\cos su]$$

$$I^{(D)}_{n+\frac{1}{2}}\quad i^{m+1}\int_0^\infty e^{i\lambda c-i\lambda\mu_1\zeta_1}J_m[\lambda\sqrt{1-\mu_1^2}\sqrt{1-\zeta_1^2}]d\lambda. \qquad \ldots \qquad (u)$$

Again, from above,

$$P_n^m(\mu)Q_n^m(\zeta)=\frac{1}{\pi}\frac{(n+m)!}{n-m)!}(-)^n\sqrt{\frac{\pi}{2D}}\,I^{(D)}_{n+\frac{1}{2}}$$

$$\int^\pi \frac{\cos mu\,du}{c-\mu_1\zeta_1+\sqrt{1\mu_1^2}\sqrt{1-\zeta_1^2}\cos u}$$

$$=\frac{1}{\pi}\frac{(n+m)!}{(n-m)!}(-)^n\sqrt{\frac{\pi}{2D}}\,I^{(D)}_{n+\frac{1}{2}}\sum_{r=0}^\infty(2r+1)\,Q_r(c)$$

$$\int_0^\pi P_r[\mu_1\zeta_1\sqrt{1-\mu_1^2}\sqrt{1-\zeta_1^2}\cos u]\cos mu\,du.$$

$$=(-)^{m+n}\frac{(n+m)!}{(n-m)!}\sqrt{\frac{\pi}{2D)}}\,I^{(D)}_{n+\frac{1}{2}}\sum_{r=m}^\infty(2r+1)\frac{(r-m)!}{(r+m)!}$$

$$Q_r(c)\,P_r^m(\mu_1\,P_n^m(\zeta_1).$$

[Introducing the well-known formula, $=(-)^{m+n}\dfrac{(n+m)!}{(n-m)!}\sqrt{\dfrac{\pi}{2D}}$

$$Q_r(c)=\sqrt{\frac{\pi}{2}}\,i^{-r-1}\int_c^\infty e^{i\lambda c}J^{(\lambda)}_{r+\frac{1}{2}}\frac{d\lambda}{\sqrt{\lambda}},\Big] \quad \dots \quad \dots \quad (v)$$

$$I^{(D)}_{n+\frac{1}{2}} \quad \sqrt{\frac{\pi}{2}}\sum_{r=m}^{\infty} i^{-r-1}(2r+1)\frac{(r-m)!}{(r+m)!}P^m_r(\mu_1)P^m_n(\zeta_1)$$

$$\int_c^\pi e^{i\lambda c}J^{(\lambda)}_{r+\frac{1}{2}}\frac{d\lambda}{\sqrt{\lambda}}.$$

Whence from (u) and (v), we may have a relation

$$e.^{-i\lambda\mu_1\zeta_1}\,J_m[\lambda\sqrt{1-\mu_1^2}\sqrt{1-\zeta_1^2} = \sqrt{\frac{\pi}{2\lambda}}$$

$$\sum_{r=m}^{\infty} i.^{-r+m}(2r+1)\frac{(r-m)!}{(r+m)!}J^{(\lambda)}_{r+\frac{1}{2}}P^m_r(\mu_1)P^m_r(\zeta_1),$$

or putting $\lambda=-kR$, $\mu_1=\cos\theta$, $\zeta_1=\cos a$, we have the formula

$$e.^{ikR\cos\theta\cos a}\,J_m[kR\sin\theta\sin a]=\sqrt{\frac{\pi}{2kR}}$$

$$\sum_{r=m}^{\infty} i.^{r-m}(2r+1)\frac{(r-m)!}{(r+m)!}J^{(kR)}_{r+\frac{1}{2}}P^m_r(\cos\theta)\,P^m_r(\cos a),$$

which, when $m=0$, reduces to Bauer's formula, which might be obtained directly by the above processes.

Sound Waves due to the Vibration of a Spheroid in the Presence of a Rigid and Fixed Spheroidal Obstacle.

Introduction.

An attempt has been made in the present paper to consider the waves resulting from a spheroid of small eccentricity, moving parallel to its axis of revolution, and the scattering of such a system, due to the presence of another rigid and fixed spheroid of small eccentricity having a common axis of revolution. The method adopted is one of successive reflections, due to Scwarz. The theory as developed by H. Poincare,* Helge Von Koch † and others of solving linear equations when the unknown quantities as also the equations are infinite in number, and a transformation-formula have been useful. Two determinants of infinite order of Von Koch's type have been solved.

We have begun with a method of obtaining the waves resulting from a small vibrating spheroid moving parallel to its axis of revolution. Though the vibratory motion of a solid of revolution is known in some form,‡ the advantages of the present method over the other consists in that the effect of Harmonic terms of higher orders can be calculated, which justifies the space devoted to it.

1. Preliminary statements.
2. Equation.
3. A method due to C. Niven.
4. Solution by infinite determinants.
5. The velocity potential.
6. A second fixed spheroid.
7. Appropriate transformation-formula.
8. Formation of equation.
9. Approximate determination of the co-efficients.
10. The velocity-potential.
11. Observation.

* Bulletin de la Societe Mathematique de France, t. 14, p. 77 ; and t. 13, p. 19.

† Acta Mathematica, Vol. 16, pp. 217-95.

‡ Dynamical Theory of Sound, H. Lamb, Art. 79, p. 290.

PART I.

Preliminary statements.

A prolate spheroid is supposed to vibrate with a velocity, $ue^{ik\epsilon t}$, (where u is small) along its axis of revolution, which is chosen as the z-axis ; (x, y, z), (r, θ, ω) and ζ, μ, ω denote the Cartesian, polar and prolate spheroidal co-ordinates of a point in space, so that we have

$$\left.\begin{aligned} x &= k.\sqrt{\zeta^2-1}\,\sqrt{1-\mu_1^2}\cos\omega = r\sqrt{1-\mu^2}\cos\omega, \quad \mu=\cos\theta, \\ y &= k.\sqrt{\zeta^2-1}\,\sqrt{1-\mu_1^2}\sin\omega = r\sqrt{1-\mu^2}\sin\omega, \\ z &= k.\mu_1\zeta \qquad\qquad = r\mu \end{aligned}\right\}\cdots\cdots$$

where $k. = a_0 e =$ the focal distance from the centre.

$(a_0 e) =$ the semi-major axis and eccentricity respectively, of the generating ellipse of the spheroid,

and $\zeta = 1/e$, over the given spheroid.

Therefore

$$\left.\begin{aligned} r^2 &= k.^2(\mu_1^2+\zeta^2-1) \\ \tan\theta &= \frac{\sqrt{1-\mu^2}}{\mu} = \frac{\sqrt{\zeta^2-1}\,\sqrt{1-\mu_1^2}}{\mu_1\zeta} \end{aligned}\right\}\cdots\cdots$$

whence

$$\left.\frac{\partial r}{\partial\zeta} = \frac{k_0{}^2\zeta}{r};\quad \frac{\partial\mu}{\partial\zeta} = \frac{\mu(1-\mu^2)}{\zeta(1-\zeta^2)},\ \text{and also}\ \frac{\partial z}{\partial\zeta} = k_0\mu_1 = \frac{r\mu}{\zeta}\right\}\cdots \quad (1)$$

Let the given spheroid be represented by

$$\left.\begin{aligned} r &= a[1+\epsilon P_2(\mu) \qquad & \mu &= \cos\theta. \\ \text{where}\quad a &= a.(1-\epsilon), & 3\epsilon &= e^2 \end{aligned}\right\}\cdots\ \cdots \quad (2)$$

The velocity-potential ϕ of the motion that would be set up due to the vibration of the spheroid, must satisfy the wave-equation

(i) $\ddot{\phi} = c^2\nabla^2\phi$, $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$, $\quad c =$ wave-velocity,

(*ii*) its first and second derivatives must be finite and continuous everywhere in the medium,

(*iii*) it must vanish as $\dfrac{e^{-ikr}}{r}$, as r tends to infinity,

(*iv*) it must satisfy the boundary-condition, $-\dfrac{\partial\phi}{\partial n}=\dfrac{\partial z}{\partial n}e^{ikct}$, on the given spheroid, dn denoting an element of normal at (x, y, z) on the surface of the spheroid.

Equations to determine co-efficients.

The motion being evidently symmetrical about the axis of revolution, we assume,

$$\phi=\sum_{n=0}^{\infty} \mathrm{R}_n(kr)^n f_n(kr)\mathrm{P}_n(\mu).\quad e^{ikct}, \qquad \dots \quad (3)$$

where the R's are constants to be determined from the boundary condition (*iv*) above, f represents the function which Prof. Lamb has introduced in his Hydrodynamics to denote diverging wave-system, satisfying the recurrence-relations*

$$\left.\begin{aligned} f'_n(x)&=-xf_{n+1}(x).\\ f_{n-1}(x)&=xf'_n(x)+(2n+1)f_n(x)\end{aligned}\right\} \qquad \dots \quad \dots$$

The element of normal at (x, y, z) on the surface $\zeta=$constant, is given by $dn=\dfrac{d\zeta}{h}$, where $1/h^2=\left(\dfrac{\partial x}{\partial\zeta}\right)^2+\left(\dfrac{\partial y}{\partial\zeta}\right)^2+\left(\dfrac{\partial z}{\partial\zeta}\right)^2.$

Hence the boundary condition (*iv*) is reduced to

$$-\frac{\partial\phi}{\partial\zeta}=u\frac{\partial z}{\partial\zeta}\quad e^{ikct}, \text{ on the spheroid.}$$

Therefore, we have

$$-\frac{u\mu r}{\zeta}=\left[\frac{\partial r}{\partial\zeta}\cdot\frac{\partial}{\partial r}+\frac{\partial\mu}{\partial\zeta}\cdot\frac{\partial}{\partial\mu}\right]$$

* Hydrodynamics, H. Lamb, 5th ed., p. 480.

$$\sum_{n=0}^{\infty} R_n(kr)^n f_n(kr) P_n(\mu), \text{ on } r = a[1+\epsilon P_2(\mu),$$

$$= k^2 k_0^{\,2}\zeta\Big[\sum_{n=0}^{\infty} R_n(kr)^{n-2} P_n(\mu)\{nf_n(kr) + krf'_n(kr)\}\Big]$$

$$+\frac{\mu(1-\mu^2)}{\zeta(1-\zeta^2)}\sum_{n=0}^{\infty} R_n(kr)^n f_n(kr) P'_n(\mu), \text{ by (1), on the spheroid}$$

or

$$-\frac{u}{k} P_1(\mu) = (ka_0.)^2 \sum_{n=0}^{\infty} R_n P_n(\mu)(kr)^{n-3}[nf_n(kr) - (kr)^2]f_{n+1}(kr)]$$

$$+\frac{\mu(1-\mu^2)}{1-\zeta^2}\sum_{n=0}^{\infty} R_n(kr)^{n-1} f_n(kr) P'_n(\mu), \text{ on the spheroid,}$$

where we have made use of

and
$$\left.\begin{aligned} k_0 &= a_0 c, \; P_1(\mu) = \mu, \\ f'_n(x) &= -xf_{n+1}(x) \end{aligned}\right\}$$

$$\sum^{\infty} R_n P_n(\mu)(ka.)^{n-1}[n\{f_n(ka.) - \epsilon(n-3)f_n(ka.)(1-P_2)$$

$$+\epsilon\overline{ka_0}^2 f_{n+1}(ka.)(1-P_2)\}$$

$$-\overline{ka_0}^2\{f_{n+1}(ka.) - \epsilon(n-1)f_{n+1}(ka.)(1-P_2) + \epsilon\overline{ka_0}^2 f_{n+2}(ka.)(1-P_2)\}]$$

$$-3\epsilon\mu(1-\mu^2)\sum^{\infty} P'_n(\mu)R_n(ka.)^{n-1}$$

$$[f_n(ka.) - \epsilon(n-1)f_n(ka.)(1-P_2) + \epsilon(ka.)^{-2}f_{n+1}(ka.)(1-P_2)]$$

where we have ignored and have yet to ignore all powers of ϵ beyond the first and used Taylor's Theorem of expansion, $\zeta = \frac{1}{e}$ and trhe ecurrence-relation $f'_n(x) = -xf_{n+1}(x)$.

Now, we may easily deduce that

$$\mu(1-\mu^2)P'_n(\mu)=\frac{n(n+1)}{2n+1}$$

$$\left[\frac{n-1}{2n-1}P_{n-2}(\mu)-\frac{n+2}{2n+3}P_{n+2}(\mu)+\frac{2n+1}{(2n-1)(2n+3)}P_n(\mu)\right],$$

and

$$P_2(x)P_n(x)=\frac{3}{2}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}P^{(x)}_{(n+2)}.$$

$$+\frac{n(n+1)}{(2n-1)(2n+3)}P^{(x)}_n+\frac{3}{2}\frac{n(n-1)}{(2n-1)(2n+1)}P^{(x)}_{(n-2)}.$$

Calculated from the formula due to Adam,* m and n being positive integers, $m \angle n$,

$$P_m(x)P_n(x)=\sum_{r=0}^{m}\frac{A_{m-r}A_rA_{n-r}}{A_{m+n-r}},\ \frac{2n+2m-4r+1}{2n+2m-2r+1}P^{(x)}_{n+m-2r},$$

where

$$A_m=\frac{(2m)!}{2m.(m!)^2},$$

whence we may write

$$-\frac{u}{k}P_1(\mu)\equiv\sum_{n=0}^{\infty}R_n(ka_0)^{n-1}[P_n(\mu)\{nf_n(ka_0)-n(n-3)\epsilon f_n(ka_0)$$

$$-\overline{ka_0}^2f_{n+1}(ka_0)+\epsilon(2n-1)\overline{ka_0}^2f_{n+1}(ka_0)-\epsilon\overline{ka_0}^4f_{n+2}(ka_0)\}$$

$$+\epsilon\{n(n-3)f_n(ka_0)-(2n-1)\overline{ka_0}^2f_{n+1}(ka_0)+\overline{ka_0}^4f_{n+2}(ka_0)\}$$

$$\left\{\frac{3}{2}\frac{n+1)(n+2)}{(2n+1)(2n+2)}P_{n+2}+\frac{n(n+1)P_n^{(\mu)}}{(2n+3)(2n-1)}+\frac{3}{2}\frac{n(n-1)P_{n-2}}{(2n+1)(2n-1)}\right\}$$

$$+3\epsilon\sum_{n=0}^{\infty}R_nf_n(ka_0)(ka_0)^{n-1}.\frac{n(n+1)}{2n+1}$$

$$\left[\frac{n+2}{2n+3}P^{(\mu)}_{n+2}-\frac{2n+1}{(2n+3)(2n-1)}P^{(\mu)}_n-\frac{(n-1)P^{(\mu)}_{n-2}}{2n-1}\right],$$

* Proc. Royal Soc., Vol. 27.

where we have ignored all powers of ϵ beyond the first

$$= \sum_{n=0}^{\infty} \frac{3\epsilon}{2} R_n (ka_0)^{n-1} \frac{(n+1)(n+2)}{(2n+3)(2n+1)}$$

$$[n(n-1)f_n(ka_0) - (2n-1)(ka_0)^2 f_{n+1}(ka_0) + (ka_0)^4 f_{n+2}(ka_0)] P_{n+2}(\mu)$$

$$+ \frac{3\epsilon}{2} \sum_{n=0}^{\infty} R_n (ka_0)^{n-1} \cdot \frac{n(n-1)}{(2n+1)(2n-1)}$$

$$[(n^2-5n-2)f_n(ka_0) - (2n-1)(ka_0)^2 f_{n+1}(ka_0)$$

$$+ (ka_0)^4 f_{n+2}(ka_0)] P_{n-2}(\mu)$$

$$+ \sum_{n=0}^{\infty} R_n (ka_0)^{n-1} P_n(\mu) \Bigg[n f_n(ka_0) \left\{ 1 - 3\epsilon \frac{(n-1)(n^2-n-4)}{(2n+3)(2n-1)} \right\}$$

$$ka_0^2 f_{n+1}(ka_0 \left\{ 1 - 3\epsilon \frac{n^2+n-1}{2n+3} \right\}$$

$$- 3\epsilon \overline{ka_0}^4 f_{n+2}(ka_0) \frac{n^2+n-1}{(2n+3)(2n-1)} \Bigg]$$

after some simplifications and collecting the Zonal Harmonics.

Hence, equating the co-efficients of the Zonal Harmonics on both sides, we have

$$-\frac{u}{k} = R_1 \left[f_1(ka_0) - \left(1 - \frac{3\epsilon}{5} \right) \overline{ka_0}^2 f_2(ka_0) - \frac{3\epsilon}{5} (ka_0)^4 f_3(ka_0) \right]$$

$$- \frac{9\epsilon}{35} (ka_0)^2 R_3 [8f_3(ka_0) + 5(ka_0)^2 f_4(ka_0) - (ka_0)^4 f_5(ka_0)],$$

and

$$0 = \frac{3\epsilon}{2} \frac{n(n-1)R_{n-2}}{(2n-1)(2n-3)}$$

$$[(n-2)(n-3)f_{n-2}(ka_0) - (2n-5)\overline{ka_0}^2 f_{n-1}(ka_0) + \overline{ka_0}^4 f_n(ka_0)]$$

$$+ \frac{3\epsilon}{2} \frac{(n+1)n+2)}{(2n+3)(2n+5)} \overline{ka_0}^4 R_{n+2}$$

$$[(n^2-n-8)f_{n+2}(ka_0)-(2n+3)\overline{ka_0}^2 f_{n+3}(ka_0)+\overline{ka_0}^4 f_{n+4}(ka_0)]$$

$$+\overline{ka_0}^2 R_n \Big[n f_n(ka_0)\left\{1-3\epsilon\frac{(n-1)(n^2-n-4)}{(2n+5)(2n-1)}\right\}$$

$$-\overline{ka_0}^2 f_{n+1}(ka_0)\left\{1-3\epsilon\frac{n^2+n-1}{2n+3}\right\}$$

$$-3\epsilon\frac{n^2+n-1}{(2n+3)(2n-1)}\overline{ka_0}^4 f_{n+2}(ka_0)\Big], \qquad \dots \quad (4)$$

where n may have all values from zero to infinity, excepting unity.

We thus have an infinity of linear equations to obtain the infinite number of R's. The theory of the solutions of such equations has been developed by H. Poincaré, Helge Von Koch and others, already alluded to, in the introduction. G. W. Hill* was the first to introduce such equations into analysis and was led to consider a determinant of infinite order, whose properties originally pointed out by Hill, were rigorously established by H. Poincaré. We proceed to find the R's by solving an infinite determinant of Koch's type.

We assume that $\overline{ka_0}^3$ is of the order of ϵ so that $\overline{ka_0}^6$ and higher powers would be ignored, while products such as $\epsilon\,\overline{ka_0}^2$ would be retained.

Let

(5) $$p_{n-2}=\frac{n(n-1)(2n+3)(2n+5)}{(n+1)(n+2)(2n-1)(2n-3)}.$$

$$\frac{(n-2)(n-3)f_{n-2}(ka_0)-(2n-5)\overline{ka_0}^2 f_{n-1}(ka_0)+\overline{ka_0}^4 f_n(ka_0)}{(n^2-n-8)f_{n+2}(ka_0)-(2n+3)\overline{ka_0}^2 f_{n+3}(ka_0)+\overline{ka_0}^4 f_{n+4}(ka_0)}\cdot\frac{1}{\overline{ka_0}^4}$$

and

(6) $$q_n=\frac{2(2n+3)(2n+5)}{3\epsilon\overline{ka_0}^2(n+1)(n+2)}.$$

$$\frac{n f_n(ka_0)\left[1-3\epsilon\frac{(n-1)(n^2-n-4)}{(2n+3)(2n-1)}\right]-\overline{ka_0}^2 f_{n+1}(ka_0)\left[1-3\epsilon\frac{n^2+n-1}{2n+3}\right]-\frac{3\epsilon(n^2+n-1)\overline{ka_0}^4 f_{n+2}(ka_0)}{(2n+3)(2n-1)}}{(n^2-n-8)f_{n+2}(ka_0)-(2n+3)\overline{ka_0}^2 f_{n+3}(ka_0)+\overline{ka_0}^4 f_{n+4}(ka_0)}$$

* Acta Math., VIII, 1886.

and also

$$(7) \quad -x\mathrm{R}_1 = \frac{35\dfrac{u}{h}}{9\epsilon\overline{ka_0}^2[8f_3(ka_0)+5\,\overline{ka_0}^2 f_4(ka_0)-\overline{ka_0}^4 f_5(ka_0)}, \text{ where}$$

x is unknown.

Then the above linear equations become

$$\left.\begin{aligned} (x+q_1)\mathrm{R}_1+\mathrm{R}_3&=0 \\ p_{n-2}\mathrm{R}_{n-2}+q_n\mathrm{R}_n+\mathrm{R}_{n+2}&=0, \quad n\neq 1 \end{aligned}\right\} \quad \ldots \quad (8)$$

We observe that the above linear equations form themselves into two independent groups, (*i*) with odd R's, (*ii*) with even R's.

The linear equations with odd R's.

(*i*) Consider the infinite determinant.

$$\begin{vmatrix} 1 & \dfrac{1}{q_1+x} & 0 & 0 & 0 \\ \dfrac{p_1}{q_3} & 1 & \dfrac{1}{q_3} & 0 & 0 \\ 0 & \dfrac{q_3}{q_5} & 1 & \dfrac{1}{q_5} & 0 \end{vmatrix}$$

This is a determinant of Von Koch's type; since p_n is at least of order $\overline{ka_0}^4$ (see 5) and q_n, of order $\dfrac{1}{ka.}$ (see 6) and we are neglecting terms of order $\overline{ka_0}^6$ or ϵ^2, the determinant is found to be absolutely convergent.* Therefore there exists value of x which would make the determinant vanish, which is obtained, as is obviously seen, as a result of step-by-step elimination of the odd R's, from the above linear equations (8) with necessary changes.

* Mod. Analysis, Whittaker and Watson, p. 37, 3rd ed.

So, x would be given by

$$\left.\begin{aligned} q_3(q_1+x) &= p_1 \\ i.e., \quad x &= -q_1+\frac{p_1}{p_3}. \end{aligned}\right\}$$

Considering the nature of orders of $\frac{p_1}{q_3}$ and q_1, we see that the second term can be neglected in comparison with the first and therefore, finally x is given by

$$x=-q_1 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \tag{9}$$

Application of a method due to C. Niven to determine x.*

We observe that the above linears equations are exactly similar in form to those obtained by C. Niven in determining the co-efficients of the assumed solution of the differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dp}{dx}\right]+\left[n(n+1)-\frac{m^2}{1-x^2}+\mu x^2\right]p=0, \ \mu, \text{ small,}$$

in a series of associated functions with Heine's definition. By following his method the above value of x is once again found.

The even R's.

As a result of step-by-step elimination of the even R's from the infinite number of linear equations, we secure the vanishing of an infinite determinant of Von Koch's type, which is found, by the principle previously quoted elsewhere, to be absolutely convergent for all values of ka_0 and ϵ within the limits of our assumption. The corresponding linear equations must therefore be identically zero and consequently the even R's are all non-existent, which is also otherwise obvious from the consideration of the nature of symmetrical motion.

The odd R's.

The value of R_7 may be calculated from and remembering that† $f_1(ka).)=\frac{1+ika.}{ka_0{}^3}e^{-ika}$

$$f_2(ka.)=-\frac{e^{-ika}}{ka_0{}^3}\left[-3-3ika.+k^2a.^2\right],$$

etc. etc. etc.

* Phil. Trans. Royal Soc., 1880.

† Hydrodynamics, H. Lamb, 5th ed., p. 480.

we find

$$R_1 = \frac{ka_0}{2} . \overline{ka_0}^2 \left[1 - \frac{18\epsilon}{5} - i\frac{k^3a^3}{6} \right], \text{ after some labour}$$

$R_3 = 0$, obviously

$$R_5 = \frac{99}{50} . \frac{R_1}{(ka_0)^2} . \frac{f_2(ka_0) - \overline{ka_0}^2 f_3(ka_0)}{-2f_5(ka_0) - 9\overline{ka_0}^2 f_6(ka_0) + \overline{ka_0}^4 f_7(ka_0)},$$

etc. etc. etc. ... ...

we observe that to our order of approximation

$$R_3 = R_5 = R_7 = \quad \dots \quad \dots \quad \dots \quad = 0,$$

so that

$$\phi = R_1(kr) f_1(kr)\, e^{ikct} P_1(\mu) \quad \dots \quad \dots \quad \dots \quad (10)$$

PART II.

The scattering in the presence of a fixed spheroid.

Let there be another fixed prolate spheroid with the same axis of revolution, having its center O' at a distance D from that of the vibrating spheroid; (r', θ', ω) and $(\zeta', \mu' \omega)$ denote the polar and prolate spheroidal co-ordinates of a point in space with respect to O' as origin, θ and θ' being in opposite sides from the Z-axis. Then,

$$\kappa'_0 \sqrt{\zeta'^2 - 1} \quad \sqrt{1 - \mu'^2_1} \;=\; r' \sqrt{1 - \mu'^2}, \qquad \mu' = \cos.\, \theta'.$$

$$\kappa'_0\, \mu'_1\, \zeta' = r'\, \mu' \quad ,$$

where $\kappa'_0 \; = a'_0 e' =$ the focal distance from the center,

$(a'_0\ e')$ = the semi-major axis and eccentricity respectively of the generating ellipse of the spheroid.

and $\zeta' = 1/e'$, over the spheroid,

where, as before, we may have,

$$\frac{\partial r'}{\partial \zeta'} = \frac{\kappa'_0{}^2 \zeta'}{r'}\ ; \quad \frac{\partial \mu'}{\partial \zeta'} = \frac{\mu'(1 - \mu'^2)}{\zeta'(1 - \zeta'^2)}, \text{ and also } \frac{\partial z'}{\partial \zeta'} = \frac{r'\mu'}{\zeta'}, \; z' \text{ being the}$$

z-co-ordinate of a point with O' as origin ... (11)

we assume that the spheroid is represented by

$$\left.\begin{aligned} r' &= a'[1+\epsilon' \, P_2(\mu')], \qquad & \mu' &= \cos.\, \theta', \\ \text{where} \qquad a' &= a'_0\,(1-\epsilon'), & 3\epsilon' &= e'^2 \end{aligned}\right\} \quad \ldots \quad (12)$$

The system of waves (10) found in Part I, would be incident on the fixed spheroid and be scattered. If ϕ_1 denote the velocity potential of the scattered system; it should evidently satisfy the condition

$$\frac{\partial}{\partial \eta}\,(\phi+\phi_1)=0, \quad \text{over the fixed spheroid.}$$

A transformation corresponding to a change of origin.

$$kr\, f_1\,(kr)\, P_1\,(\mu)$$

$$= i\; P_1(\mu)\frac{e^{-ikr}}{kr}.\; f_1\,(ikr)\; *$$

$$= -i\; P_1(\mu)\; P_1\left(\frac{d}{dikr}\right)\frac{e^{-ikr}}{kr}\; \dagger$$

$$= -i\; P_1\left(\frac{\partial}{\partial ikz}\right)\frac{e^{-ikr}}{kr},$$

$$= i\, P_1\left(\frac{\partial}{\partial ikz'}\right)\left[\sum_{s=0}^{\infty}(2s+1)(kD)^s f_s(kD)(kr')^s\psi_s(kr')P_s(\mu')\right]^{\ddagger},$$

$$r' \angle D,$$

$$\because \frac{\partial}{\partial z} = -\frac{\partial}{\partial z'}$$

$$= i\sum_{s=0}^{\infty} i^{-s}(2s+1)(kD)^s f_s(kD)P_1\left(\frac{\partial}{\partial ikz'}\right)P_s\left(\frac{\partial}{\partial ikz'}\right)\frac{\sin kr'}{kr'}, \;\S$$

$$= i\sum_{s=0}^{\infty} i^{-s}(kD)^s f_s(kD)\left[(s+1)P_{s+1}\left(\frac{\partial}{\partial ikz'}\right)+sP_{s-1}\left(\frac{\partial}{\partial ikz'}\right)\right]\frac{\sin kr'}{kr'},$$

* Introducing Stoke's notation; the two f's are certainly distinct.

† Theory of Sound, Vol. II, p. 259, Lord Rayleigh.

‡ Hydrodynamics, H. Lamb, 5th ed., p. 486.

§ Theory of Sound, Vol. II, p. 263, Lord Rayleigh and Hydrodynamics, 5th ed. p. 479, H. Lamb.

$$= - \sqrt{\frac{\pi}{2kr'}} \sum_{s=0}^{\infty} (kD)^s f_s(kD) \Big[(s+1) J_{s+\frac{1}{2}}^{(kr')} P_{s+1}^{(\mu')} - s J_{s-\frac{1}{2}}^{(kr')} P_{s-1}^{(\mu')} \Big],$$

$$= \sum_{s=0}^{\infty} (kD)^s f_s(kD) \Big[s(kr')^{s-1} \psi_{s-1}(k'r) P_{s-1}(\mu') - (s+1)(kr')^{s+1} \psi_{s+1}(kr') P_{s+1}(\mu') \Big]^*$$

$$= \sum_{s=0}^{\infty} (kr')^s \psi_s(kr') P_s(\mu') [(s+1)(kD)^{s+1} f_{s+1}(kD) - s(kD)^{s-1} f_{s-1}(kD)]$$

$$= \sum_{s=0}^{\infty} A(1,s,D)(kr')^s \psi_s(kr') P_s(\mu'), \qquad \ldots \quad (13)$$

where $A(1,S,D)=(s+1)(kD)^{s+1} f_{s+1}(kD) - s(kD)^{s-1} f_{s-1}(kD)$,... (14)

The determination of ϕ_1.

Observing that ϕ_1 corresponds to a diverging wave-system, we assume

$$\phi_1 = \sum_{p=0}^{\infty} S_p (kr')^p f_p(kr') P_p(\mu') \, e^{ickt}, \qquad \ldots \quad (14a)$$

where the S's are constants to be determined from the boundary condition over the fixed spheroid.

Therefore, we have

$$0 = \frac{\partial}{\partial \zeta'} (\phi + \phi_1), \text{ over the fixed spheroid}$$

$$= \left[\frac{\partial r'}{\partial \zeta'} \cdot \frac{\partial}{\partial r'} + \frac{\partial \mu'}{\partial \zeta'} \cdot \frac{\partial}{\partial \mu'} \right] (\phi + \phi_1), \quad \ldots \quad \ldots \quad \ldots$$

$$= \left[\frac{k'_0{}^2 \zeta'}{r'} \cdot \frac{\partial}{\partial r'} + \frac{\mu'(1-\mu'^2)}{\zeta'(1-\zeta'^2)} \cdot \frac{\partial}{\partial \mu'} \right] \left[\sum_{p=0}^{\infty} \Big\{ S_p (kr')^p f_p(kr') P_p(\mu') + R_p A(1, p, D)(kr')^p \psi_p(kr') P_p(\mu') \Big\} \right]$$

by the above transformation formulas (13) and (11),

* Hydrodynamics, 5th ed., p. 479, H. Lamb.

whence, proceeding as in Part I, we get the following equation to determine the co-efficients,

[p having all values from zero to infinity],

$$0=S_p(ka'_0)^2\Big[\ pf_p(ka'_0)\left\{1-3\epsilon'\ \frac{(p-2)(p^2+p-1)+p+1}{(2p+3)(2p-1)}\right\}$$

$$-(ka'_0)^2f_{p+1}(ka'_0)\left\{1-6\epsilon'\ \frac{p(p^2+p-1}{(2p+3)(2p-1)}\right\}$$

$$-3\epsilon'\ \frac{p^2+p-1}{(2p+3)(2p-1)}\cdot(ka'_0)^4f_{p+2}(ka'_0)\Big]$$

$$+R_1A(1,\ p,\ D)(ka'_0)^2\left[\text{an expression similar to the above in which } f \text{ is replaced by } \psi\right]$$

$$+\frac{3\epsilon'}{2}R_1A(1,\ p,\ D)\ \frac{p(p-1)}{(2p-1)(2p-3)}\Big[\ (p-2)^2\psi_{p-2}(ka'_0)$$

$$-2(p-2)\overline{ka'_0}^2\psi_{p-1}(ka'_0)+\overline{ka'_0}^4\psi_p(ka'_0)\Big]$$

$$+\frac{3\epsilon'}{2}\ S_{p-2}\cdot\ \frac{p(p-1)}{(2p-3)(2p-1)}\left[\text{a similar expression in which } \psi \text{ is replaced by } f\right]$$

$$+\frac{3\epsilon'}{2}S_{p+2}\overline{ka'_0}^4\ \frac{(p+1)(p+2)}{(2p+3)(2p+5}\ [(p^2-6)f_{p+2}(ka'_0)$$

$$-2(p+2)\overline{ka'_0}^2f_{p+3}(ka'_0)+(ka'_0)^4f_{p+4}(ka'_0)]$$

$$+\frac{3\epsilon'}{2}R_1A(1,\ p.\ D)\ \frac{(p+1)(p+2)}{(2p+3)(2p+5)}\ (ka'_0)^4\ [\text{an expression similar}$$

to the above in which f is replaced by ψ] ... ... (15)

where powers of ϵ' beyond the first have been ignored.

The determination of the S's.

Thus, we again get an infinite number of linear equations to determine the infinite number of the (*i*) even and (*ii*) the odd S's.

Let

$$\left.\begin{aligned} a_p &= \text{The co-efficient of } S_p \\ b_{p-2} &= \dots \quad \dots \quad \dots S_{p-2} \\ c_{p+2} &= \dots \quad \dots \quad \dots S_{p+2} \\ d &= \text{The sum of the three terms involving } R_1 \end{aligned}\right\} \quad \dots \quad (16)$$

Then the above equation may be written in the form

$$b_{p-2}S_{p-2} + a_pS_p + c_{p+2}S_{p+2} + d_p = 0. \quad \dots \quad (17)$$

Determination of the even S's.

From (17) we have

$$\left.\begin{aligned} a_0S_0 + c_2S_2 + d_0 = 0, \quad p=0, \\ b_{p-2}S_{p-2} + a_pS_p + c_{p+2}S_{p+2} = 0, \quad p=2, 4, 6 \dots \textit{ad inf.} \end{aligned}\right\} \quad \dots \quad (18)$$

$$\left.\begin{aligned} \text{Put} \quad d_0 = xS_0, \quad &\text{where } x \text{ is unknown} \\ dp = \lambda_{p+2}s_{q\cdot 2} \quad ,, \quad &\lambda_{p+2} \quad ,, \end{aligned}\right\} \quad \dots \quad (19)$$

The equation (18) can now be written in the form

$$\left.\begin{aligned} (x+a_0)S_0 + c_2S_2 = 0, \quad p=0 \\ b_{p-2}S_{p-2} + a_pS_p + (\lambda_{p+2} + c_{p+2})S_{p+2} = 0, p=2, 4, 6 \dots \textit{ad inf.} \end{aligned}\right\} \quad \dots \quad (20)$$

Consider the infinite determinant

$$\begin{vmatrix} 1 & \dfrac{c_2}{x+a_0} & 0 & 0 & 0 & \dots & \dots \\ \dfrac{b_0}{a_2} & 1 & \dfrac{\lambda_4 + c_4}{a_2} & 0 & 0 & \dots & \dots \\ \dots \; 0 & \dfrac{b_2}{a} & 1 & \dfrac{\lambda_6 + a_6}{a_4} & 0 & & \dots \\ & & \dots\dots & & \dots & \dots & \dots \\ \dots & & \dots\dots & & \dots & \dots & \dots \end{vmatrix}$$

which is obtained as a result of step-by-step elimination of the S's from the equations (20) with necessary changes. This is a determinant of Koch's type and is absolutely convergent if

$$\frac{b_0 c_2}{a_2(x+a_0)}+\frac{\lambda_4+c_4}{a_4}\frac{b_2}{a_2}+\ \dots\ \dots\ \dots+\frac{\lambda_{p+2}+c_{p+2}}{a_{p+2}}\frac{b_p}{a_p}+\dots\dots$$

is absolutely convergent, which would be fulfilled if we assume that

$\frac{\lambda_{p+2}+c_{p+2}}{a_{p+2}}.\frac{b_p}{a_p}$, $p=2, 4, 6, \dots$ *ad inf.*, may be neglected to our order of approximation. ... (21)

On this supposition, there exists value of x which would make the determinant vanish and the value of x is accordingly given by

$$x+a.=\frac{b_0 c_2}{a_2}.$$

It may be easily verified that the right-hand member of this equation is negligible, to our order of approximation, *i.e.*, assuming ka_0 and ka'_0 to be of the same order in magnitude, so that, finally

$$x=-a.=\overline{ka'}_0{}^4[f_1(ka'_0)+\epsilon'(ka'_0)^2 f_2(ka'_0)], \qquad \dots \quad (22)$$

whence remembering that*

$$\psi_n(x)=\frac{2^n.n!}{(2n+1)!}\left[1-\frac{x^2}{2(2n+3)}+\frac{x^4}{2.4.(2n+3)(2n+5)}-\dots\dots\dots\right]$$

$$S_0=\frac{d_0}{x}=+\frac{R_1}{x}(ka'_0)^3 A(1,0,D)=-\frac{ua_0}{6}(ka_0)^2(ka'_0)^3 A(1,0,D) \text{ by}$$

19) ... (23)

where

$$\text{by (14)}\quad A\,(1,0,D)=\frac{e^{-ikD}}{\overline{k}D^2}(1+ikD), \qquad \dots \quad (24)$$

And it may be easily seen that

$$S_2=0, \text{ obviously}$$

and $S_p=0$, $p=4, 6, \dots\dots$ *ad. inf.*, to our order of approximation ... (25)

* Hydrodynamics, 5th ed., p. 478, H. Lamb.

It may be remarked that the assumption (21) is equivalent to the assumption, $s_2=0$, and if we calculate the values of $\lambda_{p,2}$ from (19) it is easy to see that the assumption (21) is sufficiently justified.

The odd S's.

Proceeding exactly in the above manner, the odd S's may be ascertained. In fact, we have

$$S_1 = + \frac{ua_0}{2\cdot 6}(ka_0)^2)(ka'_0)^3 A(1, 1, D). \quad \ldots\ldots\ldots\ldots \quad (26)$$

where

$$\text{by (14) } A(1, 1\ \ D) = \frac{e^{-ikD}}{kD^3}\left[3+3ikD - 2kD^2\right] \quad \ldots \quad (27)$$

$$S_3 = 0, \quad \text{obviously,}$$

$S_p=0$, $p=5, 7, 9, \ldots$ *ad. inf.*, to our order of approximation... (28)

The velocity-potential.

Thus the velocity-potential of the motion is given by (10 and 14*a*)

$$\phi+\phi_1 = [R_1 kr f(kr)P_1(\mu) + S_0 f_0(kr') + S_1 kr' f_1(kr')P_1(\mu')]e^{ikct},$$

where

R_1, S_0, S_1, have been ascertained above.

Observation.

(*i*) To our order of approximation, no further reflection is necessary, for, if a second reflection is taken, we would get an equation similar to (15), in which R_1 is replaced by a combination of S_0 and S_1 (which are of the fifth order) and consequently the new co-efficients would be of a higher order than what we require.

(*ii*) We observe that according to our assumption with regard to the magnitudes of ka_0 and ka'_0 and to our order of approximation, the fixed spheroid may be regarded as a sphere, there being no effect of ϵ or ϵ' on S_0 and S_1, as also the vibrating one, so far as the reflections are concerned.

ON THE FREE VIBRATION OF A GAS IN A RIGID CYLINDER OF ELLIPTIC SECTION

Introduction.

The vibration of a gas in a rigid cylinder of circular * section had long been considered; the vibrations within a spherical envelope has been discussed by Sir George Stokes; † Chree ‡ has considered the motion between two concentric spheres. An attempt has been made here to discuss the free vibration of a gas in a rigid cylinder of elliptic section with the help of Mathieu-functions. With the success of E. L. Ince § in having the elliptic cylinder functions of the second kind corresponding to Mathieu-functions, the consideration of the vibration of a gas within two rigid confocal elliptic cylinders is no longer difficult; special cases have been considered and the corresponding frequencies determined.

1. Preliminary statements.
2. A short history of the researches on Mathieu's equation.
3. The velocity-potential.
4. Conditions of continuity.
5. Complete elliptic cylinder; particular cases.
6. Confocal elliptic cylinders; particular cases.

Preliminary Statements.

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the cross-section of the elliptic cylinder which is supposed to be infinitely long so that the motion will be a two-dimensional one, as being transverse. The gas being assumed to be friction-less, if ϕ denote the velocity-potential of the motion within the contemplated space, it must satisfy the wave-equation.

$$(i)\quad \ddot{\phi}=c^2\nabla_1{}^2\phi,\ \nabla_1{}^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\ c, \text{ the wave-velocity.}$$

* Duhamel, Liouville, Jour. Math, Vol. 14, p. 36, 1849.

† Theory of Sound, Vol. II, p. 264, Art. 331, Lord Rayleigh.

‡ Messenger of Mathematics, Vol. XV, p. 20, 1886.

§ Proc. Edin. Math. Soc., Vol. 33, p. 2.

(*ii*) Its first and second derivative must be finite and continuous everywhere within the space.

(*iii*) It must satisfy the boundary condition $\frac{\partial \phi}{\partial n}=0$, over the ellipse, dn denoting an element of the normal at any point (x, y) on the boundary.

If $\frac{2\pi}{\kappa c}$ denote the period of vibration and ϕ varies as $e^{i\kappa c t}$, the wave-equation $\ddot{\phi}=c^2\nabla_1{}^2\phi$, would be reduced to $(\nabla_1{}^2+\kappa^2)\phi=0$. (x)

Transforming to elliptic co-ordinates by the scheme

$x+iy=h\cosh(\xi+i\eta)$, so that $x=h\cosh\xi\cos\eta$

$y=h\sin h\xi\sin\eta$

$h=$ The distance of a focus from the center,

and $\xi=$ constant are confocal ellipses,

the partial differential equation (x) reduces to

$$\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}+h^2\kappa^2(\cosh^2\xi-\cos^2\eta)\,\phi=0 \qquad \ldots \quad (y)$$

Assuming a solution $\phi=F(\xi)G(\eta)$, in a product-form, where $F(\xi)$ is a function of ξ only add $G(\eta)$, a function of η only, and substituting in (y), we have,

$$\frac{\partial^2 F}{\partial\xi^2}+(h^2k^2\cosh^2\xi-A)\,F=0, \text{ and } \frac{\partial^2 G}{\partial\eta^2}+(h^2k^2\cos^2\eta)\,G=0, \quad (z)$$

where A is a constant,

each of which reduces, by substituting $\xi=iz$ and $\eta=z$ respectively, to the Mathieu-equation,

$$\frac{\partial^2 u}{\partial z^2}+(a+16q\cos 2z)\,u=0, \qquad \ldots \qquad \ldots \quad (t)$$

where $a=A-\frac{h^2k^2}{2}, q=-\frac{h^2k^2}{32}$. $\ldots$ (u)

A short history of the researches on Mathieu's equation.

M. Mathieu * had first to face the above differential equation (t) in attempting to obtain the vibratory motion of an elliptic membrane. Mathieu's differential equation is a special case of a general differential equation which arises in G. W. Hill's † (who was the first to introduce infinite determinants into analysis) investigation of the motion of the Lunar Perigee and in Adam's ‡ determination of the motion of the Lunar Node. In most of the physical (as distinguished from astronomical) problems, only periodic solutions or what are called Mathieu functions, as Prof. Whittaker calls them, are wanted. These are infinite in number and are written, introducing Prof. Whittaker's notations, as

$$\left.\begin{array}{lllll} ce_0(z),\ ce_1(z),\ ce_2(z), & \ldots & \ldots & \ldots & ce_n(z), \\ \quad se_1(z),\ se_2(z), & \ldots & \ldots & \ldots & se_n(z), \end{array}\right\}$$

The solutions of the above differential equation (t) are integral functions of z, therefore Mathieu-functions may be expressed in Fourier series and would contain cosines or sines of even multiples of z or of odd multiples of z, and therefore are symmetrical or anti-symmetrical functions of $\frac{\pi}{2}-z$, a general proof of which has been given by Jefferys. § E. L. Ince || has carried on researches on the solution of Mathieu's differential equation. An account of Mathieu-functions has been given by S. Goldstein ¶ in the Transactions of the Cambridge Philosophical Society, Vol. XXIII.

We observe that Mathieu's differential equation is linear with a periodic co-efficient which is a single-valued function of the independent variable. Floquet ** has given an analytical investigation of the nature of a general solution of the equation of the type, which, previous to the publication of his theory was otherwise perceived by astronomers from circumstantial inferences. According to Floquet's

* Liouville xii, 1868, cours de Physique Mathematique, 1873, p. 122.

† Acta Math., Vol. VIII, 1886.

‡ Monthly Notices, R.A.S., XXXVIII, p. 43.

§ Proc. Lond. Math. Soc., Vol. XXIII, p. 441, 1924.

|| Proc. Edn. Math. Soc., Vol. 33.

¶ Trans. Camb. Phil. Soc., Vol. X, 1927.

** Mod. Analysis, 3rd Ed., p. 412.

theory, the general solution of Mathieu's equation will be of the form,

$$u = Ae^{\mu z}\psi(z) + Be^{-\mu z}\psi(-z),$$

where

(*i*) $\psi(z)$ and $\psi(-z)$ are periodic functions having the same period as the co efficient of the differential equation,

(*ii*) μ is a constant, being a definite function of the constants of the original differential equation, and

(*iii*) A and B are arbitrary constants of the solution.

When the constants of the Mathieu-equation are such that $\mu=0$, the above solution fails to give the general solution, the function $\psi(-z)$ ceasing to be distinct from the function $\psi(z)$; one solution is purely periodic and two distinct methods for obtaining the corresponding second solution, called elliptic cylinder functions of the second kind, which are non-periodic, have been advanced by Juce * who introduces, for representing them, the notions as suggested by Prof. Whittaker,

$$\left.\begin{array}{llll} \mathrm{in}_0, (z), \mathrm{in}_1(z), \mathrm{in}_2(z), & \dots & \dots & \mathrm{in}_n(z). \\ Jn_1(z), Jn_2(z), & \dots & \dots & Jn_n(z), \quad \dots \end{array}\right\}$$

Prof. Whittaker † himself has given a very powerful and elegant method, *viz.*, the method of change of parameters, as he calls it, for obtaining a general solution [reducing as special cases to the Mathieu-functions $ce_1(z)$, $se_1(z)$] in Floquet's form ; the method can be extended to provide for expressions ‡ reducing to any required Mathieu-function as a special case. The general solutions of Mathieu's equation have also been analytically investigated by Dougall. ||

* Proc. Edin. Math. Soc., 33.

† ,, ,, ,,

‡ Proc Edin. Math. Soc. Vol. 33, Young.

|| ,, ,, ,, ,, ,, XXIV, 1916, pp. 4-23.

,, ,, ,, ,, ,, XLI, 1923, pp. 26-48.

,, ,, ,, ,, ,, XLIV, 1926, pp. 57-71.

The value of ϕ.

From the above discourses, we have, since G (η) must be a purely periodic function, as particular solutions,

$$\phi = F(\xi)G(\eta)e^{ikct} = ce_n(\eta, q)[Ace_n(i\xi, q) + Bin_n(i\xi, q)]e^{ikct},$$

$$Se_n(\eta, q)[Ase_n(i\xi,q)BJn_n(i\xi,q)]e^{ikct},$$

where n may have all values from zero to infinity. ... (p)

*Condition of continuity.**

To apply the above solution to the problem of the vibrations of a gas contained in an infinite rigid cylinder of elliptic section, we see that ϕ, $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$ vary in a continuous manner throughout the whole area of the elliptic section. Consider two points m and m' symmetrical with respect to the right line joining the foci and very close to one another, ξ of the two points would be the same and very small, but n would be equal and of opposite sings.

We have

$$\left.\begin{aligned} \frac{\partial \phi}{\partial \eta} &= -\frac{\partial \phi}{\partial x} h \cosh \xi \sin \eta + \frac{\partial \phi}{\partial y} h \sinh \xi \cos \eta, \\ \frac{\partial \phi}{\partial \xi} &= \frac{\partial \phi}{\partial x} h \sinh \xi \cos \eta + \frac{\partial \phi}{\partial y} h \cosh \xi \sin \eta \end{aligned}\right\}$$

Therefore, if ξ is zero or infinitely small, we may write,

$$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &= -\frac{1}{h \cosh \xi \sin \eta} \frac{\partial \phi}{\partial n}, \\ \text{and} \quad \frac{\partial \phi}{\partial y} &= \frac{1}{h \cosh \xi \sin \eta} \frac{\partial \phi}{\partial \xi}. \end{aligned}\right\}$$

ϕ, $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$ of the two points m and m' must differ by an infinitely small amount from another and would be equal to one another when

* M. Mathieu, Cours de Physique Mathematique, pp. 76-77, 139.

their ξ vanishes. Therefore representing generally ϕ by $\phi\ (\xi, n)$, we have

$$\left.\begin{aligned}&\phi(o, \eta)=\phi(o,-\eta),\\ &\frac{\partial}{\partial\eta}\phi(o, \eta)=\frac{\partial}{\partial\eta}\phi(o,-\eta);\quad \frac{\partial}{\partial\xi}\phi(o, \eta)=-\frac{\partial}{\partial\xi}\phi(o,-\eta);\end{aligned}\right\}$$

as the condition of continuity ... ... (B)

Appropriate solution for a complete cylinder.

Applying the above conditions of continuity (B) to the particular solutions (p), we see that the co-efficients associated with the second solutions vanish and therefore the appropriate solutions for the particular problem we have in view, are

$$\left.\begin{aligned}\phi=&\,\mathrm{A}.ce_n(\eta.\ q)ce_n(i\xi,q)e^{ikvt},\\ &\mathrm{B}.Se_n(\eta.\ q)Se_n(i\xi,\ q)e^{ikvt},\end{aligned}\right\}$$

Reducing the boundary condition (iii) *viz.*, $\frac{\partial\phi}{\partial n}=0$, to $\frac{\partial\phi}{\partial\xi}=0$,

since $dn=\frac{\partial\xi}{\partial h}$, where $\frac{1}{h_1^2}=\frac{\partial(x,\ y)}{\partial(\xi,\ \eta)}$, we have

$C'e_n(i\xi_0,q)=0$ and $S'e_n(i\xi_0,q)=0$, to determine the frequency of vibration, $\xi=\xi_0$ denotes the elliptic boundary, ... (u)

Particular cases.

We suppose that q is so small that its squares and higher powers may be neglected, then choosing,

(i) $C'e_1(i\xi_\circ, q)=0$, we have,*

Sinh $\xi_0+3q\sinh 3\xi_0=0$, where $\cosh\xi_0=\frac{1}{e}$, $e=$ eccentricity of the boundary

whence $q=-\frac{h^2k^2}{32}=-\frac{\sinh\xi_\circ}{3\sinh 3\xi_\circ}=-\frac{1}{3}.\frac{e^2}{4-e^2}$.

$$\therefore\quad ka=4\sqrt{\frac{2}{3}}.\ \frac{1}{\sqrt{4-e^2}},$$

* E. L. Ince. Proc. Edin. Math. Soc., Vol. 33.

The corresponding frequency is given by

$$f=\frac{ka}{2\pi}=\frac{c}{2\pi a}.ka.=\frac{2c}{\pi a}.\sqrt{\frac{2}{3(4-e^2)}}. \qquad \dots \quad (v)$$

(*ii*) Choosing $S'e_1(i\xi.q)=0$, we have,

$$\text{Cosh}\ \xi.+3q\ \cosh 3\xi.=0, \qquad \text{whence,}$$

$q=-\frac{h^2k^2}{3z}=-\frac{1}{3}\ \frac{e^2}{4-3e^2}$, and the frequency is given by

$$f=\frac{2c}{\pi a}\sqrt{\frac{2}{3(4-3e^2)}}, \qquad \dots \qquad \dots \quad (w)$$

Two confocal elliptic cylinders.

In the case of the vibrations of a gas enclosed between two infinite confocal elliptic cylinders, the appropriate particular solutions of the two-dimensional wave-equation $\ddot{\phi}=c^2\nabla_1^2,\phi$, are

$$\phi=Ce_n(\eta,q)[Ace_n(i\xi,q)+Bin_n(i\xi,q)],\ e^{ikct},$$

$$Se_n(\eta,q)[Ase_n(i\xi,q)+BJn_n(i\xi,q)]e^{ikct}$$

where n may have all values from zero to infinity and $in_n\ (i\xi,q)$, $Jn_n\ (i\xi,q)$ are elliptic cylinder-functions of the second kind, corresponding to the Mathieu-functions, as obtained by E. L. Ince.

Defining the internal and external boundaries by $\xi=\xi_0$ and $\xi=\xi_1$ respectively, the boundary conditions are $\frac{\partial\phi}{\partial\xi}=0$, for $\xi=\xi.$ and ξ_1, whence, choosing the solution $\phi=ce_n(\eta\ q)[Ace_n(i\xi,q)+Bin_n(i\xi,q)]$

we have
$$\left.\begin{aligned} Ac'e_n(i\xi.,q)+Bi'n_n(i\xi.,q)=0 \\ Ac'e_n(i\xi_1,q)+B(i'n_n(i\xi_1,q)=0 \end{aligned}\right\},$$

to determine the ratio A : B and q and therefore the frequency.

Eliminating A and B, we have,

$$\begin{vmatrix} C'e_n(i\xi.,q) & i'n_n(i\xi.,q) \\ C'e_n(i\xi_1,q) & i'n_n(i\xi_1,q) \end{vmatrix}=0, \qquad \dots \qquad \dots \quad (q)$$

Particular Cases.

We retain, as before, only the first power of q and taking $n=1$, we have,*

$$Ce_1(i\xi_\cdot, q) = -i[\sinh \xi_\cdot + 3q \sinh 3\xi_\cdot].$$

$$i'n_1(i\xi_\cdot, q) = -8q(\xi_\cdot \sinh \xi_\cdot + \cosh \xi_\cdot) + \cosh \xi_\cdot + 3q \cosh 3\xi_\cdot,$$

with similar expressions for $\xi = \xi_1$.

Hence from the determinantal equation (q), we have,

$$0 = (\sinh \xi_\cdot + 3q \sinh 3\xi_0)[\cosh \xi_1 + 3q \cosh 3\xi_1 - 8q(\xi_1 \sinh \xi_1 + \cosh \xi_1)]$$
$$- (\sinh \xi_1 + 3q \sinh 3\xi_1)[\cosh \xi_\cdot + 3q \cosh 3\xi_\cdot - 8q(\xi_\cdot \sinh \xi_\cdot + \cosh \xi_\cdot)],$$

whence, rejecting the square of q, simplifying and remembering that $q = -\dfrac{h^2k^2}{32}$,

we get

$$kh = \frac{2\sqrt{2}}{\sqrt{3 \cosh \xi_\cdot \cosh \xi_1 \cosh (\xi_\cdot + \xi_1) + \dfrac{2 \sinh \xi_\cdot \sinh \xi_1 (\xi_1 - \xi_\cdot)}{\sinh (\xi_1 - \xi_\cdot)} - 2}} \quad \ldots \quad \ldots \quad (o)$$

The corresponding frequency is given by

$$f = \frac{kc}{2\pi} = \frac{c}{2\pi h}(kh), \text{ where } kh \text{ is given above} \quad \ldots \quad (r)$$

Special Cases.

(a) When $\xi_1 - \xi_\cdot$ is negligibly small, so that we have a cylindrical sheet of elliptic section, the corresponding frequency is given by (o) and (r),

$$f = \frac{c}{2\pi h} \cdot \frac{2\sqrt{2} \cdot e^2}{\sqrt{6 - e^2}}, \; e = \text{eccentricity of the internal boundary.}$$

* † *Ibid.*

(*b*) An interesting case occurs when we choose the internal boundary to be $\xi_{\cdot}=0$. This corresponds to the case of vibration of a gas enclosed in an elliptic cylinder with a rigid plane partition stretching from one focus to the other. This effect of the partition is to render possible a difference of pressure on its two sides. If there is no difference of pressure, the partition can be removed and the vibration would be the same as in a complete elliptic cylinder. If, however, a difference of pressure exists, the corresponding frequency for the particular case we have considered above is given by

$$f=\frac{[illegible]}{\pi h}\sqrt{\frac{2}{3-2e^2}}\;;\quad e=\text{eccentricity of the external boundary.}$$

On Associated Legendre Functions and Spherical Harmonics.

Introduction.

Observing a difference in forms of $\frac{d}{dx}\ \mathrm{P}^s{}_n(x)$, a reduction-formula of the associated functions has been obtained. Application of the formula have been made in evaluating certain definite integrals involving associated functions of different degrees and orders, such as $\int_{-1}^{1} \mathrm{P}^m{}_n(x)\mathrm{P}^s{}_q(x)dx$ and $\int_0^1 \mathrm{P}^m{}_n(x)\mathrm{P}^s{}_t(x)dx$, the forms of the differential co-efficients of the solution of Laplace's equation and of the partial differential equation $(\Delta^2+k^2)\mathrm{V}=0$, have been derived and applied to spheroidal Harmonics; with its aid, a law of operating by $\mathrm{D}^2=a^2\frac{\partial^2}{\partial x^2}+b^2\frac{\partial^2}{\partial y^2}+c^2\frac{\partial^2}{\partial z^2}$, upon the elementary solution of Laplace's equation has been obtained and applied to ellipsoidal Harmonics.

1. Two forms of $\frac{d}{dx}\ \mathrm{P}^s{}_n(x)$.
2. A reduction formula.
3. Applications; the evaluation of certain definite integrals.
4. Forms of the differential co-efficients of the elementary solutions of Laplace's equation and of $(\Delta^2+k^2)\mathrm{V}=0$.
5. Applications; a law of operation by $\mathrm{D}^2=a^2\frac{\partial^2}{\partial x^2}+b^2\frac{\partial^2}{\partial y^2}+c^2\frac{\partial^2}{\partial z^2}$.

Two Forms of $\frac{d}{dx}\ P^s{}_n(x)$.

We have

$$\frac{d}{dx}\ \mathrm{P}^s{}_n(x)=\frac{\mathrm{P}_n{}^{s+1}(x)}{\sqrt{1-x^2}}-s.\ \frac{x}{1-x^2}\ .\ \mathrm{P}^s{}_n(x).$$

But

$$\frac{P_n^{s+1}(x)}{\sqrt{1-x^2}} = (1-x^2)^{\frac{s}{2}} \frac{d^s}{dx^s}\left[\frac{d}{dx} P_n(x)\right]$$

$$= (2n-1)P^s_{n-1}(x) + (2n-5)P^s_{n-3}(x) + (2n-9)$$

$$P^s_{n-5}(x) + \ldots\ldots\ldots\ldots$$

(by def. and a well-known formula)

$$= \sum_{r=0} (2n-4r-1)P^s_{n-2r-1}(x), \ldots\ldots\ldots\ldots\ldots\ldots$$

[The series is to be continued so long as the degree of any associated function is not less than its order.]

And

$$\frac{P^s_n(x)}{1-x^2} = (1-x^2)^{\frac{s-2}{2}} \cdot \frac{d^{s-2}}{dx^{s-2}} \left[\frac{d^2}{dx^2} P_n(x)\right]$$

$$= (1-x^2)^{\frac{s-2}{2}} \cdot \frac{d^{s-2}}{dx^{s-2}} \left[(2n-1)(2n-3)P_{n-2}(x)\right.$$

$$\left. + 2(2n-7)(2n-3)P_{n-4}(x) + \ldots + \ldots\right],$$

[by using the formula,

$$\frac{d^m}{dx^m} P_n(x) = \sum_r \frac{(r+m-1)!}{r!(m-1)!} \cdot \frac{1\cdot 3\cdot 5\ldots.(2n-2r-1)}{1\cdot 3\cdot 5\ldots.(2n-2r-2m+1)}$$

$$(2n-2m-4r+1)P_{n-m-2r}(x).]$$

$$= \sum_{r=1} r(2n-4r+1)(2n-2r+1)P^{s-2}_{n-2r}(x).$$

Whence since

$$xP^s_n(x) = \frac{(n-s+1)}{2n+1} P^s_{n+1}(x) + \frac{n+s}{2n+1} P^s_{n-1}(x),$$

we have

$$\frac{xP^s_n(x)}{1-x^2} = \sum_{r=0} A_r P^{s-2}_{n-2r-1}(x)$$

where

$$A_r = (2n-4r-1)[(n-r)(2r+1)-s+1],$$

Thus,

$$\frac{d}{dx} P^s_n(x) = \sum_{r=0} (2n-4r-1)[P^s_{n-2r-1}(x)$$

$$-s\{(n-r)(2r+1)-s+1\}P^{s-2}_{n-2r-1}(x)].$$

The Second Form.

Differentiating Legendre's equation

$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right]+n(n+1)P_n(x)=0.$ $(s-1)$ times with respect to x and then multiplying by $(1-x^2)^{\frac{s}{2}-1}$, we have.

$$(1-x^2)^{\frac{s}{2}}\frac{d^{s+1}}{dx^{s+1}}P_n(x)-2sx(1-x^2)^{\frac{s}{2}-1}\frac{d^s}{dx^s}P_n(x)$$

$$+(n+s)(n-s+1)(1-x^2)^{\frac{s}{2}-1}\frac{d^{s-1}}{dx^{s-1}}P_n(x)=0, \qquad \dots \quad (1)$$

Again

$$\frac{d}{dx}P^s{}_n(x)=(1-x^2)^{\frac{s}{2}}\frac{d^{s+1}}{dx^{s+1}}P_n(x)$$

$$-sx(1-x^2)^{\frac{s}{2}-1}\frac{d^s}{dx^s}P_n(x), \qquad \dots \quad (2)$$

whence by subtraction,

$$\frac{d}{dx}P^s{}_n(x)=sx(1-x^2)^{\frac{s}{2}-1}\frac{d^s}{dx^s}P_n(x)$$

$$-(n+s)(n-s+1)(1-x^2)^{\frac{s}{2}-1}\frac{d^{s-1}}{dx^{s-1}}P_n(x)$$

$$=\frac{sx}{1-x^2}P^s{}_n(x)-(n+s)(n-s+1)(1-x^2)^{\frac{s-2}{2}}\frac{d^{s-2}}{dx^{s-2}}\left[\frac{d}{dx}P_n(x)\right]$$

$$=s\sum_{r=0}A_rP^{s-2}{}_{n-2r-1}(x)-(n+s)(n-s+1)$$

$$\sum_{r=0}(2n-4r-1)P^{s-2}{}_{n-2r-1}(x)$$

$$=\sum_{r=0}(2n-4r-1)\left[s(n-r)(2r+1)-n(n+1)\right]P^{s-2}{}_{n-2r-1}(x).$$

Reduction of Ferrer's Associated Legendre Functions.

By observing the difference in forms, as given above, of $\frac{d}{dx} P^s{}_n(x)$, a formula will be obtained to express a Ferrer's Associated Legendre function of any integral degree and order as a sum of a finite number of Associated Legendre functions of an order reduced by an even number. Thus a Ferrer's function can be expressed as a sum of a finite number of Zonal Harmonics when the order of the function is even.

We shall use the notation $T_n{}^m(x)$ to represent a Ferrer's associated Function of degree n and order m, while for brevity P'_n and P_n^{s-1} will be taken to denote $\frac{d}{d_x} P_n(x)$ and $\frac{d^{s-1}}{d_x^{s-1}}$ respectively.

We have

$$T_n^{(x)} = (1-x^2)^{\frac{s}{2}-k} (1-x^2)^k P_n{}^{(s)} \quad \dots \quad (p)$$

where k is a positive integer.

Now differentiate Legendre's equation

$\frac{d}{dx}[(1-x^2)P'_n] + n(n+1) P_n = 0$, $(s-2)$ times and get

(*i*) $(1-x^2)P_n^s - 2(s-1)xP_n^{s-1} + (n-s+2)(n+s-1)P_n^{s-2} = 0$

Again differentiate

$(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0$, $(s-1)$ times and

$P'_{n+1} - P'_{n-1} = (2n+1)P_n$, $(s-2)$ times and get

$$(ii) \begin{cases} (n+1)P_{n+1}^{s-1} = (2n+1)[xP_n^{s-1} + (s-1)P_n^{s-2}] - nP_{n-1}^{s-1}, \\ (2n+1)P_n^{s-2} = P_{n+1}^{s-1} - P_{n-1}^{s-1} \end{cases}$$

Subtract $(s-1)$ times the latter from the former, we have,

(*iii*) $(n-s+2)P_{n+1}^{s-1} = (2n+1)xP_n^{s-1} - (n+s-1)P_{n-1}^{s-1}$

Hence from (*i*), (*ii*) and (*iii*) we get,

$$(1-x^2)P_n^s = -\frac{(n-s+2)(n-s+1)}{2n+1} P_{n+1}^{s-1} + \frac{(n+s)(n+s-1)}{2n+1} P_{n-1}^{s-1} \dots (r)$$

Now multiply by $(1-x^2)$ and apply (x) to the right-hand side

$$(1-x^2)^2 P_n^s = \frac{(n-s+4)!}{(n-s)!} \cdot \frac{P_{n+2}^{s-2}}{(2n+3)(2n+1)}$$

$$-2\frac{(n+s)(n+s-1)(n-s+1)(n-s+2)}{(2n+3)(2n-1)} P_n^{s-2}$$

$$+\frac{(n+s)!}{(n+s-4)!} \frac{P_{n-2}^{s-2}}{(2n+1)(2n-1)} \qquad \ldots \quad (y)$$

From this the law can be deduced ; the general term in the expression for $(1-x^2)P_n^s$ is $K_{n,s,k,r} \; P_{n+k-2r}^{s-u}$, where

$$K_{n,s,k,r} = (-)^{k+r} . C_r^k . 2^k . \frac{(n+s)!}{(n-s)!} . \frac{(n-s+2k-2r)!}{(n+s-2r)!} .$$

$$\frac{(2n-2r)!\,(n-r+k)!}{(n-r)!\,(2n+2k-2r+1)!} (2n+2k-4r+1), \qquad \ldots \quad (z)$$

The Reduction.

The desired reduction can now be effected ; since from (p) and (z)

$$\underset{n}{\overset{s}{T}}(x) = (1-x^1)^{\frac{s-2k}{2}} \sum_{r=0}^{k} K_{n,s,k,r} \cdot \frac{d^{s-2k}}{dx^{s-2k}} P_{n+k-2r}^k ,$$

and $$P_{n+p-2x}^k = \sum_p C_p^{k+p-1} \cdot \frac{1.3.5\ldots(2n+2k-4r-2p-1}{1.3.5\ldots(2n-4r-2p+1)}$$

$$(2n-4r-4p+1) P_{n-2r-2p},$$

it follows that

$$\overset{s}{T}_n(x) = \sum_{r=0}^{k} K_{n,s,k,r} \sum_{n} \overset{k+p-1}{C_p} \cdot \frac{1.3.5\ldots(2n+2k-4r-2p-1)}{1.3.5\ldots(2n-4r-2p+1)}$$

$$(2n-4r-4p+1) \; \underset{n-2r-2p}{T(x)^{s-2k}} \qquad \ldots \quad (k)$$

The result corresponding to $k=1$, was obtained by Adams * by a different method. The observation made in the beginning is from (k) quite clear and requires no further comment.

The results corresponding to $k=2$, as deduced from the formula

$$\overset{s}{T}_n(x)=(n-s+1)(n-s+2)(n-s+3)(n-s+4)\overset{s-4}{T}_n(x)$$

$$-4(s-2)\sum_p (2n-4p+1)\left[n^2-n(2ps-2p-1)+(s-1)\{(s-2)+(p-1)(2p+1)\}\right]\overset{s-4}{T}_{n-2p}(x)$$

Application of the Formula (k).

The evaluation of the definite Integrals

$$\int_{-1}^{1}\overset{m}{T}_n(x)\,\overset{p}{T}_q(x)\,dx \text{ and } \int_{0}^{1}\overset{m}{T}_n(x)\,\overset{p}{T}_q(x)\,dx.$$

The formulae

$$\left.\begin{aligned}&\int_{-1}^{1}\overset{m}{T}_n(x)\,\overset{m}{T}_q(x)\,dx=0,\ n\neq q\\ &\int_{-1}^{1}\left[\overset{m}{T}_n(x)\right]^2 dx=\frac{2}{2n+1}\cdot\frac{(n+m)!}{(n-m)!}\end{aligned}\right\} \quad \ldots \quad (s)$$

where n, m, q are positive integers, are well-known. With the aid of the formula (k) given above, it is possible to evaluate the definite integrals quoted above. It has been found in the case of the first integral that if the orders of two associated functions be both odd, or both even or equal, the integral has usually a non-zero value. If, however, the orders are both odd or both even and if the degree of the function having the greater order is less than that of the other, or if the degrees are one odd and the other even, the integral vanishes. The second integral can be treated in a similar manner.

An example of the methods employed and results obtained is given below.

* Collected Scientific Papers, 2, 412; *cf.* also 2, 376.

Let us suppose that m and p are both even or both odd and $m>p$, so that $m-p$ is even $(=2k$, say). Then by the formula (k), since $m-p=2k$, we have,

$$\int_{-1}^{1} \mathrm{T}_n^m(x)\, \mathrm{T}_q^p(x)\, dx = \sum_{r=0}^{k} k_{n,s,k,r} \sum_{s}^{k+s-1} \mathrm{C}_s . \frac{1\cdot3\cdot5\ldots(2n+2k-4r-2s-2)}{1\cdot3\cdot5\ldots(2n-4r-2s+1)}$$

$$(2n-4r-2s+1)\int_{-1}^{1} \mathrm{T}^p_{n-2r-2s}(x)\, \mathrm{T}_q^p(x)\, dx. \qquad \ldots \quad (r)$$

From (s) and (v), it follows, when m and p are both even or both odd, that,

$$\int_{-1}^{1} \mathrm{T}_n^m(x)\, \mathrm{T}_q^p(x)\, dx \quad =0, \quad q>n.$$

$$=0, \quad [n \text{ even and } q \text{ odd or } vice\ versa].$$

$$=(-)^k . \frac{2}{2n+1} . \frac{(n+p)\,!}{(n-m)\,!}, \quad n=q$$

$$= 4\frac{(q+p)\,!}{(q-p)\,!}\left[\sum_{s=0}^{r,k}(-)^{k-s} . \mathrm{C}_s^k . \mathrm{C}_{k-1}^{r+k-s-1} . \quad (2n+2k-4s+1)\right.$$

$$\left.\times \frac{(n-p-2s)\,!(n+m)\,!(2n-2s)\,!(n-s+k)\,!(n-r-s)\,!(2n+2k-2r-2s-1)\,!}{(n+m-2s)\,!(n-m)\,!(n-s)\,!(n-r-s+k-1)\,!(2n-2r-2s+1)\,!(2n+2k-2s+1)\,!}\right].$$

if $q=n-2r$ in the series (r);

[The summation is taken over all values of s from m to r, if $r<k$; and over all values from zero to k, if $r>k$.]

Ellipsoidal Harmonics in Polar Co-ordinates by Direct Differentiation.

A Recurrence-formula.

We know

$$\mathrm{P}_n^m(x) = (-i)^m \frac{(n+m)\,!}{n\,!\pi} \int_0^{\pi} [x+\sqrt{x^2-1}\cos\phi]^n \cos m\phi d\phi$$

Therefore,

$$nxP_n^m(x)+(1-x^2)\frac{d}{dx}P_n^m(x)$$

$$=\frac{(n+m)!}{(n-1)!}\cdot\frac{(-i)^m}{\pi}\cdot\int_0^\pi [x+\sqrt{x^2-1}\cos\phi]^{n-1}\cos m\phi d\phi$$

$$=(n+m)\,P_{n-1}^m(x) \qquad \ldots \quad (a)$$

The following formulas would be useful, *

(i) $(n-m+1)\,P_{n+1}^m(x) \quad =(2n+1)\,x\,P_n^m(x)-(n+m)\,P_{n-1}^m(x)$

(ii) $(2n+1)\,\sqrt{1-x^2}\;P_n^{m-1}(x) \;=\; P_{n+1}^m(x)-P_{n-1}^m(x)$

(iii) $(2n+1)\sqrt{1-x^2}\;P_n^{m-1}(x) \quad =-(n-m)(n-m+1)\,P_{n+1}^m(x)$

$$+(n+m)(n+m+1)\,P_{n-1}^m(x).$$

Forms of Differential Co-efficients of Laplace's Equation.

An elementary solution of Laplace's equation is of the form

$$\begin{matrix} r^n \\ r^{-n-1}\end{matrix}\;P_n^m(\mu)\;\begin{matrix}\cos\\ \sin\end{matrix}\;m\phi,$$

in polar co-ordinates and obviously

$$\left[\frac{\partial}{\partial x},\;\frac{\partial}{\partial y},\;\frac{\partial}{\partial z}\right]\begin{matrix} r^n \\ r^{-n-1}\end{matrix}\;P_n^m(\mu)\;\begin{matrix}\cos\\ \sin\end{matrix}\;m\phi,$$

is a solution of Laplace's equation and therefore can be expressed in terms of the elementary solution as follows:—

Let

$$x=r\sin\theta\cos\phi;\quad y=r\sin\theta\sin\phi;\quad z=r\cos\theta;\quad \mu=\cos\theta,$$

* Collected Scientific Papers, Adams, p. 248 (*i*) and (8) and p. 262 (*ii*).

Then we may have

$$\left[\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right](r;\ \mu;\ \phi)=\sin\theta\cos\phi.\ \sin\theta\sin\phi,\ \mu$$

$$-\frac{\sin\theta\mu\cos\phi}{r},\ -\frac{\sin\theta\mu\sin\phi}{r},\ \frac{1-\mu^2}{r};$$

$$-\frac{\sin\phi}{r\sin\theta},\quad \frac{\cos\phi}{r\sin\theta},\quad 0.$$

Now

$$\frac{\partial}{\partial z}\left[r^n\ P_n^m(\mu)\ {}^{\cos}_{\sin}\ m\phi\right]=\left[n\mu\ P_n^m(\mu)+(1-\mu^2)\frac{d}{d\mu}\ P_n^m(\mu)\right]r^{n-1}\ {}^{\cos}_{\sin}\ m\phi$$

$$=(n+m)r^{n-1}\ P_{n-1}^m(\mu)\ {}^{\cos}_{\sin}\ m\phi,\ \text{by the recurrence}$$

formula (a) ... (p)

Again we know

$$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}=r\frac{\partial}{\partial r}$$

and ... (iv)

$$x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}=\frac{\partial}{\partial \phi}$$

Assuming

$$V=r^n\ P_n^m(\mu)\ e^{im\phi},$$

$$\frac{\partial v}{\partial z}=(n+m)r^{n-1}\ P_{n-1}^m(\mu)\ e^{im\phi},\quad \text{by } (p)$$

whence from (iv)

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=r^n\ e^{im\phi}\left[n\ P_n^m(\mu)-(n+m)\mu\ P_{n-1}^m(\mu)\right]$$

and

$$x\frac{\partial v}{\partial y}-y\frac{\partial v}{\partial x}=\frac{\partial v}{\partial \phi}=imr^n\ P^m(\mu)\ e^{im\phi}.$$

Therefore

$$\sqrt{1-\mu^2}\,\frac{\partial v}{\partial x}=\frac{r^{n-1}}{2}\Big[e^{i(m+1)\phi}\Big\{(n-m)\,P_n^m(\mu)-(n+m)\mu\,P'_{-1}\Big\}$$

$$+e^{i(m-1)\phi}(n+m)\Big\{P_n^m(\mu)-\mu P_{n-1}^m\Big\}\Big] \quad \ldots \quad (v)$$

and

$$\sqrt{1-\mu^2}\,\frac{\partial v}{\partial y}=i\frac{r^{n-1}}{2}\Big[e^{i(m-1)\phi}(n+m)\Big\{P_n^m(\mu)-\mu P_{n-1}^m(\mu)\Big\}$$

$$-e^{i(m+1)\phi}\Big\{(n-m)\,P_n^m(\mu)-(n+m)\mu\,P_{n-1}^m(\mu)\Big\}\Big]\ldots \quad (vi)$$

$$\text{But}\quad \overset{m}{P}(\mu)-\mu\overset{m}{P}(\mu=\frac{n+m-1}{2n-1}\Big[\overset{m}{P}(\mu)-\overset{m}{P}(\mu)\Big]$$

$$=(n+m-1)\sqrt{1-\mu^2}\;\overset{m-1}{P}(\mu)\,.\quad\text{by }(ii) \qquad (vii)$$

$$\text{and }(n-m)\,\overset{m}{P}(\mu)-(n+m)\mu\,\underset{n-1}{\overset{m}{P}}(\mu)=\frac{1}{2n-1}\Big[(n-m)(n-m-1)$$

$$\underset{n}{\overset{m}{P}}(\mu)-(n+m)(n+m-1)\,\underset{n-2}{\overset{m}{P}}(\mu)\Big]=-\sqrt{1-\mu^2}\;\underset{n-1}{\overset{m+1}{P}}(\mu)\,.\quad\text{by }(iii)$$

$$(viii)$$

we may also observe

$$n\sqrt{1-\mu^2}\;\underset{n}{\overset{m}{P}}(\mu)-\mu\sqrt{1-\mu^2}\;\frac{d}{d\mu}\underset{n}{\overset{m}{P}}(\mu)\mp\frac{mP_n^m(\mu)}{\sqrt{1-\mu^2}}$$

$$=\frac{1}{\sqrt{1-\mu^2}}\Big[(n\mp m)\underset{n}{\overset{m}{P}}(\mu)-\mu\Big\{n\mu\underset{n}{\overset{m}{P}}(\mu)+(1-\mu^2)\frac{d}{d\mu}\underset{n}{\overset{m}{P}}(\mu)\Big\}\Big]$$

$$\frac{1}{\sqrt{1-\mu^2}}\Big[(n\mp m)\,\underset{n}{\overset{m}{P}}(\mu)-(n+m)\,\mu\underset{n-1}{\overset{m}{P}}(\mu)\Big]$$

$$\left.\begin{aligned}&=-\;\overset{m+1}{P}(\mu)\,,\ \text{in the first case, by }(viii)\\ &\text{or }=(n+m)(n+m-1)\;\underset{n-1}{\overset{m-1}{P}}(\mu)\,,\ \text{in the second case by }(vii)\end{aligned}\right\}\quad (ix)$$

Hence substituting in (v) and (vi) and taking the real and imaginary parts on both sides, we have,

$$\frac{\partial}{\partial x}\left[r^n P_n^m(\mu) \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\right] = \frac{r^{n-1}}{2}$$

$$\left[-P_{n-1}^{m+1}(\mu) \begin{matrix}\cos\\ \sin\end{matrix}\, \overline{m+1}\,\phi + (n+m)(n+m-1)\, P_{n-1}^{m-1}(\mu) \begin{matrix}\cos\\ \sin\end{matrix}\, \overline{m-1}\,\phi\right]$$

$$\ldots \quad \ldots \quad \ldots \quad (V)$$

and $$\frac{\partial}{\partial y}\left[r^n P_n^m(\mu) \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\right] = \mp \frac{r^{n-1}}{2}\left[P_{n-1}^{m+1}(\mu) \begin{matrix}\sin\\ \cos\end{matrix}\, \overline{m+1}\,\phi\right.$$

$$\left. + (n+m)(n+m-1)\, P_{n-1}^{m-1}(\mu) \begin{matrix}\sin\\ \cos\end{matrix}\, \overline{m-1}\,\phi\right] \quad \ldots \quad \ldots \quad (v).$$

(p), (q), (v), are the forms we have sought.

In the particular case, when $m=0$, we have,

$$\left[\frac{\partial}{\partial x}\,,\ \frac{\partial}{\partial y}\right]\left(r^n P_n(\mu)\right) = \mp r^{n-1}\, P_{n-1}^{1}(\mu) \begin{matrix}\cos\\ \sin\end{matrix}\, \phi, \quad \ldots \quad (s)$$

Since $$P_n^{-m}(\mu) = (-)^m. \frac{(n-m)!}{(n+m)!} \,.\, P_n^m(\mu).$$

We may similarly have the forms of $\left[\frac{\partial}{\partial x}\,,\ \frac{\partial}{\partial y}\,,\ \frac{\partial}{\partial x}\right]$

$$\frac{P_n^m(\mu)}{r^{n+1}} \begin{matrix}\cos\\ \sin\end{matrix}\, m\,\phi.$$

In fact, we have,

$$\left[\frac{\partial}{\partial x}\,,\ \frac{\partial}{\partial y}\,;\ \frac{\partial}{\partial z}\right] \frac{P_n^m(\mu)}{r^{n+1}} \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi$$

$$= \pm\tfrac{1}{2}\left[(n-m+1)(n-m+2)\,\frac{P_{n+1}^{m-1}(\mu)}{r^{n+2}} \begin{matrix}\cos\\ \sin\end{matrix}\, \overline{m-1}\,\phi \mp\right.$$

$$\left.\frac{P_{n+1}^{m+1}(\mu)}{r^{n+2}} \begin{matrix}\cos\\ \sin\end{matrix}\, \overline{m+1}\,\phi \quad -(n-m+1)\,\frac{P_{n+1}^{m}(\mu)}{r^{n+2}} \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\right] \ldots \quad (t)$$

and $$\left[\frac{\partial}{\partial x}\,,\ \frac{\partial}{\partial y}\right]\frac{P_n(\mu)}{r^{n+2}} = -\,\frac{P^1_{n+1}(\mu)}{r^{n+2}} \begin{matrix}\cos\\ \sin\end{matrix}\,\phi. \quad \ldots \quad \ldots \quad (u)$$

We observe that the sth differential co-efficient of any elementary solution of the above forms can be expressed in terms of the relevant solid harmonics by the repeated application of the above formula.

Thus, $\left[\frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial x_2} \right] r^n P_n^m(\mu) \frac{\cos}{\sin} m\phi$

$$= \frac{r^{n-2}}{4} \left[\mp P_{n-2}^{m+2}(\mu) \frac{\cos}{\sin} \overline{m+2}\, \phi - 2(n+m)(n+m-1)\, P_{n-2}^{m}(\mu) \frac{\cos}{\sin} m\phi \right.$$

$$\left. \mp \frac{(n+m)!}{(n+m-4)!} P_{n-2}^{m-2}(\mu) \frac{\cos}{\sin} \overline{m-2}\, \phi \right] \quad \ldots \quad \ldots \quad (v)$$

$$\left[\frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial x^2} \right] \frac{P_n^m(\mu)}{r^{n+1}} \frac{\cos}{\sin} m\phi$$

$$= \frac{1}{4} \left[\mp \frac{P_{n+2}^{m+2}(\mu)}{r^{n+3}} \frac{\cos}{\sin} \overline{m+2}\, \phi - 2(n-m+1)(n-m+2) \frac{P_{n-2}^{m}(\mu)}{r^{n-3}} \frac{\cos}{\sin} \right.$$

$$\left. m\phi \mp \frac{(n-m+4)!}{(n-m)!} \frac{P_{n+2}^{m-2}(\mu)}{r^{n+3}} \frac{\cos}{\sin} \overline{m-2}\, \phi \right] \quad \ldots \quad \ldots \quad (w)$$

and

$$\frac{\partial^2}{\partial z^2} \left[r^n P_n^m(\mu) \frac{\cos}{\sin} m\phi \right] = (n+m)(n+m-1)\, r^{n-2} P_{n-2}^{m}(\mu) \frac{\cos}{\sin} m\phi,$$

$$\ldots \quad \ldots \quad \ldots \quad (o)$$

$$\frac{\partial}{\partial z^2} \left[\frac{P_{n+1}^m(\mu)}{r^{n+1}} \frac{\cos}{\sin} m\phi \right] = (n-m+1)(n-m+2) \frac{P_{n+2}^m(\mu)}{r^{n+3}} \frac{\cos}{\sin}$$

$$m\phi, \quad \ldots \quad \ldots \quad \ldots \quad (m)$$

and so on.

Forms of the Differential Co-efficients of the Elementary Solutions of $(\nabla^2+k^2)V=0$.

It would be convenient to take $k=1$. The elementary solutions are of the form $r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(r)\, P_n^m(\mu) \frac{\cos}{\sin} m\phi$, or introducing Prof. Lamb's

rotation,* $r^n\psi_n(r)\ P_n^m(\mu) \begin{matrix}\cos\\ \sin\end{matrix} m\phi$, where ψ_n satisfies the recurrence formulae,*

$$\left.\begin{aligned} \psi'_n(x) &= -x\psi_{n+1}(x), \\ \psi_{n-1}(x) &= x\psi'(x)+(2n+1)\psi_n(x). \end{aligned}\right\} \quad \cdots \quad \cdots \quad \text{(M)}$$

Obviously the differential co-efficients are solutions of $(\nabla^2+1)V=0$, and therefore can be expressed in terms of the elementary solutions as follows:—

$$\frac{\partial}{\partial x}\left[r^n\psi_n(r)\ P_n^m(\mu) \begin{matrix}\cos\\ \sin\end{matrix} m\phi\right]$$

$$=r^{n-1}\left[\psi_n(r)\left\{n\mu\, P_n^m(\mu)+(1-\mu^2)\frac{d}{d\mu}P_n^m(\mu)\right\}+r\psi'(r)\mu\, P_n^m(\mu)\right] \begin{matrix}\cos\\ \sin\end{matrix} m\phi.$$

$$=\frac{r^{n-1}}{2n+1}\left[(n+m)\, P_n^m(\mu)\left\{r\psi'_n(r)+(2n+1)\psi_n(r)\right\}+(n-m+1)r\psi'_n(r)\ P_{n+1}^m(\mu)\right] \begin{matrix}\cos\\ \sin\end{matrix} m\phi, \text{ by } (i) \text{ and } (v)$$

$$=\frac{1}{2n+1}\left[(n+m)r^{n-1}\psi_{n-1}(r)\ P_{n-1}^m(\mu)-(n-m+1)r^{n+1}\psi_{n+1}(r)\ P_{n-1}^m(\mu)\right] \begin{matrix}\cos\\ \sin\end{matrix} m\phi.$$

$$\frac{\partial}{\partial x}\left[r^n\psi^n(r)\ P_n^m(\mu) \begin{matrix}\cos\\ \sin\end{matrix} m\phi\right]$$

$$=r^{n-1}\left[\left\{n\psi_n(r)+r\psi'_n(r)\right\}\left\{\sqrt{1-\mu^2}\ P_n^m(\mu)-\psi_n(r)\mu\sqrt{1-\mu^2}\frac{d}{d\mu}P_n^m(\mu)\right\} \begin{matrix}\cos\\ \sin\end{matrix} m\phi\cos\phi+\frac{mP_n^m(\mu)}{\sqrt{1-\mu^2}}\psi_n(r) \begin{matrix}\sin\\ \cos\end{matrix} m\phi\sin\phi\right]$$

$$=\frac{r^{n-1}}{2}\left[\psi_n(r)\left\{n\sqrt{1-\mu^2}\ P_n^m(\mu)-\mu\sqrt{1-\mu^2}\frac{d}{d\mu}P_n^m(\mu)\mp\frac{mP_n^m(\mu)}{\sqrt{1-\mu^2}}\right\} \begin{matrix}\cos\ \overline{m-1}\,\phi \\ \sin\ \overline{m-1}\,\phi \\ \cos\ \overline{m-1}\,\phi \\ \sin\ \overline{m-1}\,\phi\end{matrix} + r\psi'_n(r)\sqrt{1-\mu^2}\ P_n^m(\mu) \begin{matrix}\cos\ \overline{m+1}\,\phi \\ \cos\ \overline{m-1}\,\phi\end{matrix}\right]$$

* Hydrodynamics, H. Lamb, 5th ed., pp. 478-80.

where both the terms corresponding to the two $\begin{matrix}\cos\\ \sin\end{matrix}$ are to be taken.

$$= \frac{r^{n-1}}{2(2n+1)}\left[\begin{matrix}\cos\\ \sin\end{matrix}\overline{m+1}\,\phi\left\{-(2n+1)\psi_n(r)P_{n-1}^{m+1}(r) \qquad + r\psi'_n(\mu)\right.\right.$$

$$\left.\left(P(\mu)_{n+1}^{m+1} - P(\mu)_{n-1}^{m+1}\right)\right\}$$

$$+ \begin{matrix}\cos\\ \sin\end{matrix}\overline{m-1}\,\phi\left\{(2n+1)(n+m)(n+m-1)\ P(\mu)_{n-1}^{m+1}\psi_n(r\right.$$

$$\left.\left.+ r\psi'_n(r,\left(\overline{n+m}.\overline{n+m-1}\right)P(\mu)_{n-1}^{m-1} - \overline{n-m+1}.\overline{n-m+2}\ P(\mu)_{n+1}^{m-1}\right)\right\}\right]$$

by (ix)

$$= \frac{r^{n-1}}{2(2n+1}\left[\begin{matrix}\cos\\ \sin\end{matrix}\overline{m+1}\ \phi\left\{r\psi'_n(r)P(\mu)_{n+1}^{m-1} \quad - \overline{r\psi'_n(r) + (2n+1)\psi_n(r)}\quad P(\mu)_{n-1}^{m+1}\right\}\right.$$

$$+ \begin{matrix}\cos\\ \sin\end{matrix}\overline{m-1}\ \phi\left\{\left(\overline{r\psi'_n(r) + (2n+1)\psi_n(r)}\right)\ (n+m)(n+m-1)P(\mu)_{n-1}^{m-1}\right.$$

$$\left.\left.-(n-m+1)(n-m+2)r\psi'_n(r)P(\mu)_{n+1}^{m-1}\right\}\right] \text{ by (M)}$$

$$= \frac{1}{2(2n+1)}\left[-\begin{matrix}\cos\\ \sin\end{matrix}\overline{m+1}\ \phi\left\{r^{n+1}\psi_{n+1}(r)\ P(\mu)_{n+1}^{m+1} \ + r^{n-1}\psi_{n-1}(r)\ P_{n-1}^{m+1}(\mu)\right\}\right.$$

$$\begin{matrix}\text{os}\\ \text{in}\end{matrix}\overline{m+1}\,\phi\left\{(n+m)(n+m-1)\,r^{n-1}\psi_{m-1}(r)\ P_{n-1}^{m-1}(\mu)\right.$$

$$\left.\left.+\ (n-m+1)(n-m+2)r^{n-1}\psi_{n+1}(r)\ P(\mu)_{n+1}^{m-1}\right\}\right].$$

Similarly, $$\frac{\partial}{\partial y}\left[r^n\psi_n(r)\,\overset{m}{\mathrm{P}}(\mu)\ \begin{matrix}\cos\\ \sin\end{matrix}\ m\phi \right]$$

$$= \mp \frac{1}{2(2n+1)}\left[\begin{matrix}\sin\\ \cos\end{matrix}\ \overline{m+1}\ \phi \left\{ r^{n+1}\psi_{n+1}(r)\overset{m+1}{\mathrm{P}}_{n+1}(\mu) + r^{n-1}\psi_{n-1}(r)\overset{m+1}{\mathrm{P}}_{n-1}(\mu) \right\}\right.$$

$$\begin{matrix}\text{in}\\ \text{os}\end{matrix}\ \overline{m-1}\ \phi \left\{ (n-m+1)(n-m+2)r^{n+1}\psi_{n+1}(r)\overset{m-1}{\mathrm{P}}_{n+1}(\mu)\right.$$

$$\left.\left. + (n+m)(n+m-1)r^{n-1}\psi_{n-1}(r)\overset{m-1}{\mathrm{P}}_{n-1}(\mu) \right\}\right]$$

The other solution, $r^n\psi_n(r)\,\overset{m}{\mathrm{P}}_n(\mu)\ \begin{matrix}\cos\\ \sin\end{matrix}\ m\phi$, may be similarly treated. Obviously, the differentiation may be effected any number of times, as before.

Application of the Differential Co-efficients of the Elementary Solution of Laplace's Equation.

Internal Prolate Spheroidal Harmonics in Polar Co-ordinates by Direct Differentiation.

W. D. Niven * has given the following expression for an internal Prolate Spheroidal Harmonic,

$$\overset{\sigma}{\mathrm{G}}_n = \left[1 - \frac{\mathrm{D}^2}{2(2n-1)} + \frac{\mathrm{D}^4}{2.4.(2n-1)2n-3)} \right.$$

$$\left. + \frac{(-)^m\mathrm{D}^{2m}}{2^m.m\,!(2n-1)2n-3)\ldots(2n-m+1)} \right] \mathrm{H}_n\,(x, y, z),$$

where $\mathrm{D}^2 = k^2\dfrac{\partial^2}{\partial z^2}$, $k^2 = c^2 - a^2$,

the greatest value of $m = n/2$, $\dfrac{n-1}{2}$, whichever is an integer and $\mathrm{H}_n(x, y, z)$ is a spherical Harmonic of degree n, of the nature of an elementary solution of Laplace's equation.

Let us suppose with Niven † that $\mathrm{H}_n(x, y, z)$ contains the longitudinal factor $\xi^\sigma + \eta^\sigma$, where $\xi = x + iy$, $\eta = x - iy$.

* Phil. Trans. Royal Soc., 1891.

† ,, ,, ,, p. 260.

With a slight change in the definition of the $H_n(x, y, z)$ which Niven has given, *i.e.*, omitting some constant factors, let us suppose that

$$H_n^\sigma(x, y, z) = r^n P_n^\sigma(\mu) \cos \sigma\phi.$$

Then the corresponding Prolate Spheroidal Harmonic in polar co-ordinates is given by

$$G_n^\sigma = \left[1 - \frac{D^2}{2(2n-1)} + \frac{D^4}{2.4.(2n-1)(2n-3)} - \dots \right] r^n P_n^\sigma(\mu) \cos \sigma\phi$$

$$= \cos \sigma\phi \left[r^n P_n^\sigma(\mu) - \frac{k^2(n+\sigma)(n+\sigma-1)}{2(2n-1)} r^{n-2} P_{n-2}^\sigma(\mu) + \dots \right]$$

by (P)

The operation would continue until the Tesseral Harmonic is converted into a sectorial one or into a Tesseral Harmonic in which the difference between its degree and order is unity.

External Prolate Spheroidal Harmonic in Polar Co-ordinates by Direct Differentiation.

W. D. Niven's formula is

$$G_n^\sigma = (-)^n \frac{2^{n+1}.n!}{(2n+1)!} H_n^\sigma \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \left[1 + \frac{D^2}{2(2n+3)} + \frac{D^4}{2.4(2n+3)(2n+5)} + \dots\dots\dots\dots \right] \frac{1}{\sqrt{x^2+y^2+z^2}}$$

where (x, y, z) denotes an external point,

$$D^2 = k^2 \frac{\partial^2}{\partial x^2}, \qquad k^2 = c^2 - a^2.$$

Niven (and later Prof. Hobson) * has shown that if $F_n(x, y, z)$ denote a Spherical Harmonic of degree n, then

$$F_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \frac{1}{r} = \frac{(-)^n.(2n)!}{2^n.n!} \cdot \frac{F_n(x, y, z)}{r^{2n+1}},$$

and therefore,

$$H_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \frac{1}{r} = \frac{(-)^n (2n)!}{2^n.n!} \frac{P_n^\sigma(\mu) \cos \sigma\phi}{r^{n+1}}$$

* Proc. Lond. Math. Soc., Vol. XXIV, 1893, p. 83.

Hence

$$G^{\sigma}(x, y, z) = \frac{2}{2n+1}\left[1 + \frac{k^2}{2(2n+3)}\frac{\partial^2}{\partial z^2} + \frac{k^4}{2.4.(2n+3)(2n+5)}\frac{\partial^4}{\partial z^4}\right.$$

$$\left. + \ldots\ldots\ldots\right]\frac{P^{\sigma}_{n}(\mu)}{r^{n+1}}\cos\sigma\phi$$

$$= \frac{2}{2n+1}\left[\frac{P^{\sigma}_{n}(\mu)}{r^{n+1}} + \frac{k^2(n-\sigma+1)(n-\sigma+2)}{2(2n+3)}\frac{P^{\sigma}_{n+2}(\mu)}{r^{n+3}}\right.$$

$$\left. + \ldots\ldots\ldots\ldots\right]\cos\sigma\phi$$

Integration of the Product of two Internal Prolate Spheroidal Harmonics over a Sphere.

Application of the above forms of the Spheroidal Harmonics.

Let G^{σ}_{n} and $G^{\sigma'}_{m}$ be two different internal prolate Spheroidal Harmonics of degrees n and m and of orders σ and σ' respectively. Expressing them in polar co-ordinates, as in the above processes, we have,

$$G^{\sigma}_{n} = \cos\sigma\phi\left[r^n P^{\sigma}_{n} - \frac{k^2(n+\sigma)(n+\sigma-1)}{2(2n-1)} r^{n-2} P^{\sigma}_{n-2}(\mu) + \ldots\ldots\right],$$

The last term being

$$\frac{(-)^{\frac{n-\sigma}{2}} . k^{n-\sigma} . (n+\sigma)!\,2(n-1)!}{2.\left(\frac{n-\sigma}{2}\right)!\left(\frac{n+\sigma}{2}\right)!\,(2n-1)!\,(2\sigma)!}\, r^{\sigma}\, P^{\sigma}_{\sigma}(\mu);$$

[$n.\sigma$ being both odd or both even]

$$\text{or } \frac{(-)^{\frac{n-\sigma-1}{2}} . k^{n-\sigma-1} . (n-1)!\,(n+\sigma)!\,(n+\sigma+1)!}{2.\left(\frac{n-\sigma-1}{2}\right)!\left(\frac{n+\sigma+1}{2}\right)!\,(2n-1)!\,(2\sigma+1)!}$$

$$. \quad r^{\sigma+1} P^{\sigma}_{\sigma+1}(\mu); \; [n.\sigma \text{ being odd. even}]$$

and

$$G_m^{\sigma'} = \cos \sigma'\phi \Big[r^m P_m^{\sigma'} (\mu) - \frac{k^2(m+\sigma')(m+\sigma'-1)}{2(2n-1)}$$

$$r^{m-2} P_{m-2}^{\sigma'} (\mu) + \ldots\ldots \Big],$$ with a similar last term.

From above it follows that when the orders σ, σ' are different, the integral $\iint G_n^\sigma \; G_m^{\sigma'} \; ds$ over a sphere, vanishes for all values of m and n. We, therefore, have to consider the integral, only when the orders are the same, *i.e.*, $\sigma = \sigma'$, and the following cases would arise,

(*i*) m, odd and n, even, and *vice versa.*
(*ii*) $m = n$.
(*iii*) n, m, both even or both odd.

In case (*i*) the integral vanishes on account of the well-known property of the associated functions $\int_{-1}^{1} P_n^s P_q^s dx = 0$, $n \neq q$.

In case (*ii*)

$$\iint [G_n^\sigma \;]^2 ds = 2\pi R^2 \Big[\frac{(n+\sigma)!}{(n-\sigma)!} \cdot \frac{R^{2n}}{2n+1} + \frac{R^{2n-4}}{2n-3} \cdot$$

$$\frac{k^4(n+\sigma)^2(n+\sigma-1)^2}{2^2.(2n-1)^2} \cdot \frac{(n+\sigma-2)!}{(n-\sigma-2)!} + \ldots\ldots \Big]$$

the last term being

$$\frac{R^{2\sigma}}{(2\sigma+1)!} \left[\frac{(n+\sigma)!^2 \; . \; (n-1)! \; k^{n-\sigma}}{2.\left(\frac{n-\sigma}{2}\right)! \left(\frac{n+\sigma}{2}\right)! \; (2n-1)!} \right]^2;$$

[n, σ being both odd or both even],

$$\text{or } R^{2\sigma+2} \cdot \frac{2\sigma+2}{(2\sigma+3)!} \left[\frac{(n-1)! \; (n+\sigma)! \; (n+\sigma+1)! \; k^{n-\sigma-1}}{2. \; (2n-1)] \left(\frac{n-\sigma-1}{2}\right)! \left(\frac{n+\sigma+1}{3}\right)} \right]^2$$

[$n.\sigma$ being odd and even].

The case (*iii*) is similar to case (*ii*) and we may have,

$$\iint G_n^\sigma\, G_m^\sigma\, ds = \pi R^2 \left[\frac{(-k^2)^{\frac{n-m}{2}}\,.\;(n+m)!\,(n-1)!}{\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!\;(2n-1)!} \right.$$

$$\frac{(n+\sigma)!\,R^{2m}}{(m-\sigma)!\,(2m+1)} + \frac{(-k^2)^{\frac{n-m+2}{2}}\,.\;(n+m-2)!\,(n-1)!}{2\left(\frac{n-m+2}{2}\right)!\left(\frac{n+m-2}{2}\right)!}$$

$$\frac{(m+\sigma)!\,(2m-2)R^{2m-1}}{(2n-1)!(2m-1)!\,(m-\sigma-2)!} \times \frac{(n+\sigma)!\,(2m-4)!}{(m+\sigma-2)!} + \ldots + \ldots\ldots + \ldots$$

$$+ \frac{(-k^2)^{\frac{n+m-2\sigma}{2}}\;(m+\sigma)!^2\,(n+\sigma)!^2\,(n-1)!}{2.\left(\frac{n-\sigma}{2}\right)!\left(\frac{n+\sigma}{2}\right)!\left(\frac{m-\sigma}{2}\right)!\left(\frac{m+\sigma}{2}\right)!}$$

$$\frac{(m-1)!\,R^{2\sigma}}{(2m-1)!\,(2n-1)!\,(2\sigma+1)!}\;;\ [n,\sigma \text{ being both odd or both even}]\;;$$

or

$$\frac{(-k^2)^{\frac{n+m-2\sigma-2}{2}}\,.\;(n+\sigma+1)!(m+\sigma+1)!\,(n+\sigma)!\;(m+\sigma!)}{2.\left(\frac{n-\sigma-1}{2}\right)!\left(\frac{m-\sigma-1}{2}\right)!\left(\frac{n+\sigma+1}{2}\right)!\left(\frac{m+\sigma+1}{2}\right)!}$$

$$\frac{(n-1)!\;(2\sigma+2)R^{2\sigma+2}(m-1)!}{(2n-1)!\,(2m-1)!\;(2\sigma+3)!}\;;\ [n,\ \sigma \text{ being odd and even}],$$

R denoting the radius of the sphere over which the integration is taken.

[A LAW OF THE FORMATION OF TERMS WHEN ANY ELEMENTARY SOLUTION OF LAPLACE'S EQUATION IS OPERATED UPON ANY NUMBER OF RELEVANT TIMES BY THE OPERATOR:

$$D^2 = a^2\frac{\partial^2}{\partial x^2} + b^2\frac{\partial}{\partial y^2} + c^2\frac{\partial^2}{\partial z^2}.\Big]$$

Take, for instance, the function $r^n P_n^m(\mu)\cos m\phi$.

It is easy to see from the formulae (r), (o) given above that

$$D^2[r^n P_n^m(\mu) \cos m\phi]$$

$$=r^{n-2}\left[P_{n-2}^{m+2}(\mu) \cos \overline{m+2}\,\phi . p+q \frac{(n+m)\,!}{(n+m-2)\,!} P_{n-2}^{m}(\mu) \cos m\phi + \right.$$

$$\left. p \frac{(n+m)\,!}{(n+m-4)\,!} P_{n-2}^{m-2}(\mu) \cos \overline{m-2}\,\phi \right],$$

where

$$p=\frac{a^2-b^2}{4} \text{ and } q=-\tfrac{1}{2}\,(a^2+b^2-2c^2).$$

Similarly,

$$D^4[r^n P_n^m (\mu) \cos m\phi]$$

$$=r^{n-4}\left[p^2 P_{n-4}^{m+4}(\mu) \cos \overline{m+4}\,\phi+2pq \frac{(n+m)\,!}{(n+m-2)\,!} P_{n-4}^{m+2}(\mu) \cos \overline{m+2}\phi \right.$$

$$+(2p^2+q^2) \frac{(n+m)\,!}{(n+m-4)\,!} P_{n-4}^{m}(\mu) \cos m\phi$$

$$+2pq \frac{(n+m)\,!}{(n+m-6)\,!} P_{n-4}^{m-2}(\mu) \cos \overline{m-2}\,\phi+p^2 \frac{(n+m)\,!}{(n+m-8)\,!}$$

$$\left. P_{n-4}^{m-4}(\mu) \cos \overline{m-4}\phi \right]$$

We have got sufficient data for generalisation ; in the operation denoted by D^{2s}, (i) there are $(2s+1)$ terms, (ii) in the first term the order is increased by the index over D, *i.e.*, $2s$, in the succeeding terms, the order is continually diminished by 2 from that of the one which precedes it until we arrive at the last term in which the order is reduced by the index over D, (iii) the degree of each of the terms is reduced by the index over D, (iv) the second term has the numerical factor $\frac{(n+m)\,!}{(n+m-2)\,!}$, in the succeeding terms, the numerator is the same as in the second while the denominator involving factorial is continually diminished by 2 from that of the term which precedes it, until the last term is arrived at in which the denominator

with the factorial is $(n+m-4s)!$ (v) The law of factors composed of p and q is somewhat complicated and requires laborious attention; after a good deal of calculation, the corresponding law is found as stated below:—

(a) The factors (involving p and q) associated with the terms equidistant from the beginning and the end are equal ; this being so,

(b) The co-efficient (involving p and q) corresponding to any term may be obtained by the following expression,

$$\sum_{m=0}^{l} \frac{s!\, p^{s-r+2m}\, q^{r-2m}}{(s-r+m)!\, m!\, (r-2m)!},$$

where

$$l = r/2,\ \frac{r-1}{2} \text{ whichever is an integer,}$$

and r may have the values, $r=0, 1, 2, 3, \ldots s$; the zero value of r corresponding to the first term, and the last value (*viz.*, s) corresponding to the middle term. A table is annexed by which the law of factors involving p and q has been found. Thus,

$$D^{2s}\left[r^n P_n^m(\mu) \cos m\phi \right],\ m>2s\ ;\ n>m$$

$$= r^{n-2s}\left[P_{n-2s}^{m+2s}(\mu) \cos \overline{m+2s}\,\phi . p^s + sp^{s-1}q \frac{(n+m)}{(n+m-2)!} \right.$$

$$P_{n-2s}^{m+2s-2}(\mu) \cos \overline{m+2s-2}\,\phi + \ldots + \ldots\ldots + \ldots\ldots + p^s . \frac{(n+m)!}{(n+m-4s)!}$$

$$\left. P_{n-2s}^{m-2s}(\mu) \cos \overline{m-2s}\,\phi \right] \ldots\ldots(u)$$

The general formula corresponding to other spherical Harmonics of this nature can similarly be worked out.

Application of the Law to Ellipsoidal Harmonics.

We may employ the above law to express ellipsoidal Harmonics corresponding to an ellipsoid with three un-equal axes, in terms of the elementary solutions of Laplace's equation, the possibility of

which is obvious. W. D. Niven's * formula (which was afterwards obtained by Prof. Hobson † by an elegant and independent method) for an internal ellipsoidal Harmonic G_n^σ of the four species is given by the scheme,

$$\left|\begin{array}{cccc} & x & yz & \\ 1 & y & zx & xyz \\ & z & xy & \end{array}\right| \Pi\left(\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}-1\right).$$

which is expressed in terms of the spherical Harmonic

$$\left|\begin{array}{cccc} & x & yz & \\ 1 & y & zx & xyz \\ & z & xy & \end{array}\right| \Pi\left(\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}\right)$$

denoted by H_n^σ, as

$$G_n^\sigma(x, y, z)=\left[1-\frac{D^2}{2(2n-1)}+\frac{D^4}{2.4.(2n-1)(2n-3)}\cdots\right.$$

$$\left.(-)^m\frac{D^{2m}}{2^m.m!(2n-1)(2n-3)\ldots(2n-2m+1)}\right]H_n^\sigma(x, y, z)$$

where $\quad D^2=a^2\frac{\partial^2}{\partial x^2}+b^2\frac{\partial^2}{\partial y^2}+c^2\frac{\partial^2}{\partial x^2}$,

and the greatest value of $m=\frac{n}{2}, \frac{n-1}{2}$ whichever is an integer.

The method may now be explained by taking a very simple case, say, when $n=2$. We know there would be all told 5 varieties of ellipsoidal Harmonics of degree 2, of which two are of the first species and three, of the third. The two values of θ in $H_2(x, y, z)$ of the first species are given by the quadratic equation,

$$(\theta+b^2)(\theta+c^2)+(\theta+c^2)(\theta+a^2)+(\theta+a^2)(\theta+b^2)=0,$$

* Phil. Trans. Royal Soc., A, 1891.

† Proc. Lond. Math. Soc., Vol. XXIV, 1893.

from which it can be noticed that one of the roots lies between $-c^2$ and $-b^2$ and another between $-b^2$ and $-a^2$. Taking one of the values, we have,

$$H_2(x,y,z)=\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}$$

$$=\frac{b^2-a^2}{3\,!(a^2+\theta)(b^2+\theta)}\,r^2P^2_2(\mu)\cos 2\phi+\frac{r^2P_2(\mu)}{c^2+\theta}$$

The corresponding ellipsoidal Harmonic is by (u),

$$G_2(x,y,z)=\left[1-\frac{D^2}{3\,!}\right]H_2(x,y,z)$$

$$=\frac{(b^2-a^2)r^2P^2_2(\mu)}{3\,!(a^2+\theta)(b^2+\theta)}\cos 2\phi+\frac{r^2P_2(\mu)}{c^2+\theta}+\frac{(a^2-b^2)^2}{3\,!(a^2+\theta)(b^2+\theta)}$$

$$+\frac{a^2+b^2-2c^2}{3\,!(c^2+\theta)}$$

Table showing the Laws (a) and (b).

$(D^2)^1$	$p \quad q \quad p$	p	q	p					
$(D^2)^2$	$p(p,q,p)$, $q(p,q.p)$, $p(p,q,p)$	p^2	$2pq$	$2p^2+q^2$	$2pq$	p^2			
$(D^2)^3$	$p^2(p,q,p)$, $2pq(p,q,p)$, $(2p^2+q^2)(p,q,p)$	p^3	$3p^2q$	$3p(p^2+q^2$	$q(6p^2+q^2)$	$3p(p^2+q^2)$	$3p^2q$	p^3	
$(D^2)^4$	$p^3(p,q,p)$, $3p^2q(p,q,p)$, $3p(p^2+q^2)$	p^4	$4p^3q$	$2p^2(2p^2+3q^2)$	$4pq(3p^2+q^2)$	$6p^4+12p^2q^2+q^4$	$4pq(3p^2+$	$p^2(2p^2+3q^2)$	$4p^3q$
$(D^2)^5$	$p^4(p,q,p)$, $4p^3q(p\ q,p)$	p^5	$5p^4q$	$5p^3(p^2+2q^2)$	$10p^2q(2p^2+q^2)$	$p(2p^4+6p^2q^2+q^4)$	$q(30p^4+22$ $+q^4)$	$(2p^4+6p^2q^2+q^4)$	$10p^2q(2p^2+q^2)$
$D^2)^6$	$p^5(p,q,p)$, $5p^4q(p,q,p)$	p^6	$6p^5q$	$3p^4(2p^2+5q^2)$	$10p^3q(3p^2+2q^2)$	$15p^2(p^4+4p^2q^2+q^4)$	$6pq(10p^4+1$ $q^4)$	$p^6+90p^4q^2$ $30p^2q^4+q^6$	$6pq(10p^4+10p^2q^2+q^4)$
$(D^2)^7$	$p^6(p,q.p)$, $6p^5q(p,q,p)$	p^7	$7p^6q$	$p^5(p^2+3q^2)$	$7p^4q(6p^2+5q^2)$	$7p^3(3p^4+15p^2q^2+5q^4)$	$7p^2q(15p^4$ $q^2+3q^4)$	$p(5p^6+30p^4q^2+15p^2q^4+q^6)$	$q(140p^6+210p^4q+42p^2q^4+q)$
$D^2)^s$		p^s	$sp^{s-1}q$	$sp^s+\frac{s(s-1)}{2!}p^{s-2}q^2$	$\frac{s(s-1)}{2!}p^{s-1}q.2+\frac{s(s-1)(s-2)}{3!}p^{s-3}q^3$	$\frac{s(s-1)}{2!}p^s+\frac{s(s-1)(s-2)}{3!}p^{s-2}q^2.3+\frac{s(s-1)(s-2)(s-3)}{4!}p^{s-4}q^4$	$\frac{s(s-1)(s-2)}{3!}p^{s-1}q.3.+\frac{s(s-1)(s-2)(s-3)}{}p^{s-3}q\ldots+\frac{s(-1)(s-2)(s-3)(s-4)}{5}p^{s-5}q^5$	$\frac{s(s-1)(s-2)}{3!}p^s.1.+\frac{s(s-1)(s-2)(s-3)}{4!}p^{s-2}q^2.6+\frac{s(s-1)(s-2)(s-3)(s-4)}{5!}p^{s-4}q^4.5+\frac{s(s-1)(s-2)(s-3)(s-4)(s-5)}{6!}p^{s-6}q^6$	$\frac{s(s-1)(s-2)(s-3)}{4!}p^{s-1}q.4.+\frac{s(s-1)(s-2)(s-3)(s-4)}{5!}p^{s-3}q^3 10+\frac{s(s-1)(s-2)(s-3)(s-4)(s-5)}{6!}p^{s-5}q^5.6+\frac{s(s-1)(s-2)(s-3)(s-4)(s-5)(s-6)}{7!}p^{s-7}q^7$

Note on the Occurrence of Sulphur in Some of the Tertiary Coals of India.—I.[1]

BY

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Introduction.

In a previous communication[2] I discussed the action of solvents on some Indian coals and the present note is concerned with a study of the sulphur found in them. The coals dealt with here are :—(*i*) Palana lignite, (*ii*) Makerwal coal, (*iii*) Mach coal, (*iv*) Dandot coal, (*v*) Jammu anthracitic coal and (*vi*) Jammu impure coal. The coals of all these seams belong to the same geological horizon—the Laki stage (middle Eocene)—a set of coals of the same geological age but ranging from lignite to anthracite. The important facts to be noted in this connection are :—

(1) that the Jammu area was subjected to severe dynamic metamorphism with the result that the seams are crushed and jointed while the coal has become friable,

(2) that in the Punjab and the Baluchistan area the coal seams were not subjected to any marked dynamic metamorphism although the beds were disturbed and

(3) that the lignites of Palana have suffered no dynamic metamorphism.

The specimens of coal under investigation were very kindly supplied by Dr. C. S. Fox of the Geological Survey of India. I am greatly indebted to him for various suggestions while carry-

[1] Read at the Indian Science Congress, Madras, 1929.

[2] Chatterjee, N N , *Quart. Journ. Geol. Min. & Met. Soc. Ind.*, Vol. II, pt. 2, pp. 89-95, 1929.

ing on my work. My thanks are also due to Prof. H. C. Das-Gupta for giving me facilities in the laboratory. The high sulphur content of the Indian Tertiary coals attracted the attention of the early workers, referred to later, but all of them determined and recorded the sulphur as total sulphur, and attributed it to the presence of iron pyrites in coal. It has now been established that there are several other forms in which sulphur occurs in coal. The present paper embodies the result of my work which is an endeavour to discover the different forms in which sulphur occurs in the coal specimens mentioned above. It may be pointed out that no systematic and detailed work regarding the sulphur content of Indian coal has been hitherto published. It is my desire to point out that the total sulphur is not the only factor to be considered. The amounts of the different sulphur forms should be determined before any coal is recommended for a particular use.

It should be mentioned in this connection that as a result of the researches of Parr and Powell [1] our knowledge regarding the sulphur content of coal has recently been greatly widened. A reference to this work shows that sulphur in coal generally occurs in the four different forms, *i.e.*, organic, pyritic, sulphate, and free. It should also be remembered that sulphur is sometimes recorded in the following manner :—

(*i*) *Fixed sulphur*—the amount of sulphur retained by the coke when the coal is carbonised. This is a very important factor to the metallurgists and iron-smelters.

(*ii*) *Volatile sulphur*—the portion of sulphur which volatilises when the coal is carbonised and consequently varies with the temperature of cabonisation. The volatile sulphur is a guide to the sulphur content of the gaseous and liquid products obtained from the coal.

(*iii*) *Non-combustible sulphur*—the amount of sulphur left in the ash when the coal is completely burnt.

(*iv*) *Combustible sulphur*—the amount of sulphur which is expelled during complete combustion of coal. When present in appreciable amount in coal this combustible sulphur corrodes the metallic structures of boilers.

[1] Parr and Powell, *Bull. Univ. of Illin.*, No. 111, pt. 7, pp. 44-45, 1919.

Description of Coal Samples.

1. *Palana Lignite.*

The coal is a resinous peaty lignite of dark brown colour which is not uniform throughout. It is very light, porous, loose and friable. It breaks into small loose fragments and soils the fingers when handled.

The specimens have a very dull appearance due to the absence of any bright bands. It has an indistinct lamination. Some specimens have one or two thin dull bands of a darker colour. The material has often numerous joints and cracks. The woody structure is largely obliterated. There are inclusions of resinous substance scattered irregularly through the specimen. The small resinous inclusions are of a yellow brown colour with a characteristic lustre. The resinous material is very brittle and can easily be reduced to powder. A few thin bands of pyrite occur in the lignite. The pyrite bands extend only over a very short distance. When the coal samples with the pyrite bands are allowed to remain exposed to air the pyrite is oxidised to sulphate which is visible to the naked eye. But such layers are not found to be numerous in the lignite specimens.

La Touche [1] (1897) noticed the presence of strings of iron pyrites but the fragments could be picked.

This lignite when carbonised gives off sulphurous fumes.

It is not a coking coal. The colour of the ash is orange red.

2. *Makerwal Brown Coal.*

The specimen has a jet black colour and shining appearance. It is appreciably heavier than the previous sample of lignite, being much more compact and far less porous. Some specimens show a waxy lustre. When struck with a hammer it breaks into cubical fragments which are compact and brittle. It slightly soils the fingers on handling. No obvious laminations or banded structure was noted in the specimens. It has inclusions of pyrite

[1] La Touche, *Rec. Geol. Surv. Ind.*, Vol. 30, p. 122, 1897.
General Report, *Geol. Surv. Ind.*, 1898-99, p. 33.

in fine lines or stringers (often more or less decomposed to ferrous sulphate). There are also small inclusions of resinous substance in patches or granules. These are soft and brittle and of yellowish brown colour and are not uniformly distributed throughout the coal but locally they have been found to be plentiful.

The coal when heated gives off sulphurous fumes. It is a good coking coal. The colour of the ash is pink.

3. *Mach Coal.*

The coal is compact, hard and lustrous. It breaks into irregular cubes with occasional development of imperfect conchoidal fracture. It is very finely laminated and has thin bright bands alternating with thicker dull bands. Several bright bands are sometimes very closely packed, otherwise they lie wide apart. The lustre is sometimes waxy. It does not soil the finger. Some of the specimens have no banded appearance and are very compact, while the specimens with banded structure are sometimes loose and friable on account of the presence of pyritic layers alternating with the coaly bands. These pyrites are often partially decomposed to sulphates. During this process they have expanded and thus the compact mass of laminated coal becomes fragile after prolonged exposure to air. This can be seen in some of the hand specimens. On splitting open one of such specimens yellowish iron pyrite could be seen in an advanced state of oxidation. There are also few resinous specks as small inclusions in the coal. Again along some joints and cracks a little cream-coloured deposit can be seen. These are found to be ankerite which was evidently deposited by percolating water.

Blanford [1] in 1880 described this Mach coal from the Bolan Pass. Greisbach [2] in 1881 while dealing with the geology of the Mach area pointed out that the coal-bearing group is greatly disturbed and folded and much shattered and jointed. He writes—"The coal is so friable and contains so much sulphur and iron pyrites that it would not be advisable to use it for smith's purposes as it would render all iron very brittle."

[1] Blanford, *Rec. Geol. Surv. Ind.*, Vol. 15, pt. 3, p. 149, 1880.

[2] Greisbach, *Mem. Geol. Surv. Ind.*, Vol. 18, p. 27, 1881.

When the coal is carbonised sulphurous fumes are freely evolved. It is a coking coal and the colour of the ash is light red.

4. *Dandot Coal.*

In appearance the coal resembles that from Mach. The coal is compact and black with brilliant lustre and has also the banded structure. The coal is much jointed and has numerous cracks. There are dull and bright bands which are very thin. It is clean and does not soil one's fingers. Numerous pyritic layers are seen alternating with the dull and bright coaly bands. There are also few larger inclusions of pyrites found occasionally in the coal. The pyrites in the weathered specimens are all converted to sulphates and the affected portions are falling to pieces and the different progressive stages in the oxidation of pyrites can be watched in the laboratory. The pyrite inclusions and disseminations are more numerous here than in the previous samples. There are inclusions of the yellowish brown resinous material scattered here and there throughout the specimen.

Oldham [1] in 1887 made an estimate about the coal reserve of the area. Simpson [2] pointed out the friable and jointed nature of the coal. He published the result of the analysis of the coal and pointed out the presence of large amount of pyrites in it. Dunstan [3] also analysed the Dandot coal and determined the total sulphur the amount of which was found to be 1·86 per cent.

When carbonised the coal yields heavy sulphurous fumes. It produces good coke. The colour of the ash is brick red showing the presence of large amount of iron oxide in the ash.

The ash on further examination was found to contain more than 85% of Fe_2O_3.

5. *Jammu Anthracitic Coal.*

The coal is jet black in colour and has greasy lustre. There are numerous thin dull and bright bands which are not very prominent. It soils the finger very slightly and appears to be of uniform texture. No inclusions of resinous substance nor any pyritic bands could be easily seen. The coal is hard and compact and appears to be much

[1] R. D. Oldham, *Ms. Report, Geol. Surv. Ind.*, 1887.
[2] R. R. Simpson, *Mem. Geol. Surv. Ind.*, Vol. 41, p. 110, 1918.
[3] Dunstan, *Rec. Geol. Surv. Ind.*, Vol. 33, part 4, p. 248, 1906.

crushed. On exposure to the air the specimen does not fall into pieces. When heated in the covered crucible the volatiles burn with little flame indicating that the coal contains very small amount of volatile matter. Sulphurous fumes are slightly present in the volatiles. The coal cakes strongly. Ash is cream-coloured showing the absence of iron in it. The coal of Jammu was originally discovered by Medlicot [1] who makes a mention of the much crushed nature of the coal. Simpson,[2] in 1904, mentioned that the coal is extremely friable but has a strongly marked foliated character. It burns with little flame or smoke. He also recorded the high percentage of sulphur in the coal.

6. *Jammu Impure Coal.*

The specimens are those of a dull coal. They are all coated with yellowish brown hydrate of iron. The coal is friable and much jointed. It is hard and compact and has very small amount of volatile matter. It has no bright or dull bands. The coal is full of limonite coatings and all the cracks and joints in the coal have been filled up with it. The coal appears consequently to be of inferior quality. Such coals with plenty of limonite coatings are to be found in some of the seams in the Jharia field and they are called " Rangi " or red-coloured coal by the local miners. These ' Rangi ' coals are picked out by hand from the loaded *tubs* and rejected. The coal has consequently very high percentage of ash which amounts to as much as 30%. Otherwise it is a low-volatile and low-moisture coal. On account of the situation of the seams in the Jammu area which was subjected to metamorphism, the coal is much shattered, and the anthracitic character has developed to some extent.

The sulphurous fumes were slightly evolved when the coal was burnt. The coal does not coke. The colour of the ash is bright red. The ash on further examination was found to be chiefly composed of iron oxides.

Determination of Sulphur—Conclusion.

In order to find out the amounts of the different forms of sulphur in the laboratory the author has followed in general the methods

[1] Medlicot, *Rec. Geol. Surv. Ind.*, Vol. 9, p. 49, 1876.

[2] Simpson, R. R., *Mem. Geol. Surv. Ind.*, Vol. 32, part 4, p. 225, 1904.

adopted by Illingworth [1] and by Parr and Powell.[2] The results obtained by the author are enumerated in the adjoining Table II and the proximate analyses and specific gravity of the abovementioned coal specimens are given in Table I.

It will be apparent from the figures that the Palana lignite and Makerwal, Mach and Dandot coals contain high percentages of volatile matter, but the Jammu coal contains little volatile matter. It is believed that on account of the metamorphism the Jammu coal has been squeezed and the volatiles and moisture were expelled to a great extent with the concentration of the fixed carbon. There must have been also the selective deposition of vegetable matter in the separate areas with the consequent variation in the composition of the different coal samples. Metamorphism has played a very important part in the conversion of the vegetable matter into different types of coal. An interesting feature about the Mach coal is the low percentage of ash (2·32%) in it whereas the total sulphur amounts to about 4%. This can be explained by the assumption that the original vegetable substance was of great purity in the first place and during the time of deposition at Mach, it was not mixed up or contaminated with large amount of impurities, nor was the coal seam after it had been deposited, traversed by percolating water charged with iron and other salts. A reference to the analytical results given in Table II shows that the Mach coal contains only a small amount of pyritic sulphur (0·4%) and that the major portion of the total sulphur is in organic form (3%). These all account for the low ash content of the Mach coal. It is also seen that the Palana lignite and Jammu coal contain a very small amount of total sulphur. It is found that practically the whole of total sulphur exists as organic sulphur in the Palana lignite, the pyritic sulphur being present only in traces. This accounts for the low ash in the Palana lignite. In Jammu coal also the major portion of the sulphur is in the organic form and the percentage of pyritic sulphur is very low. The coals of Dandot, Makerwal and Mach are very rich in total sulphur. The volatile sulphur in the Palana lignite and Jammu coal is present only in very small amount, but it runs very high in other coals of the series. It was pointed out that the Jammu area suffered more from severe

[1] Illingworth, S. R., *Analysis of Coal and its By-products*, Colliery Guardian, London, pp. 60-62, 1921.

[2] Parr and Powell, *Bull. Univ. Illin.*, No. 111, p. 44, 1919.

dynamo metamorphism than the other areas. The writer thinks that probably on account of this metamorphic effect on the coal seams, they have been much crushed with the result that some of the volatile sulphur have been expelled together with the moisture. About the Palana lignite the low volatile sulphur does not require any explanation as the lignite itself does not contain more than 1·1% of total sulphur. The composition of the vegetable 'mother substance' may also be responsible for this variation.

Another important feature about the sulphur content of the abovementioned coal specimens is that only a minute fraction of total sulphur remains in the ash whereas the major portion of sulphur is expelled during complete combustion of coal showing that pyrites were also more or less completely decomposed. The sulphate sulphur is present in the specimens only in minute quantities. It should be mentioned in this connection that particular care was always taken to select fresh specimens of coal for the chemical analysis and that the weathered specimens were always rejected as the amount of sulphate sulphur would in the weathered specimen be too high and the pyritic sulphur would be correspondingly too low. The percentage of combustible sulphur is very high in Makerwal, Mach and Dandot coals showing that they cannot be efficiently burnt on the grates of boilers. This combustible and not the total sulphur should be the most important factor to the coal consumers for their use.

In conclusion the author again emphasises on the importance of the sulphur study and draws the attention of the mining engineers and coal consumers to the fact that in order to have a proper idea regarding the nature of the sulphur in coals the different sulphur forms should be accurately determined and that they should be one basis in the valuation of the different kinds of coal.

TABLE I

Proximate Analyses and Specific Gravity.

	I Palana Lignite	II Makerwal Coal	III Mach Coal, Baluchistan	IV Dandot Coal	V Jammu Anthracitic Coal	VI Jammu Impure Coal.
Moisture	17·66	3·78	10·83	5·09	0·83	1·31
Volatile matter	41·22	49·79	40·57	43·41	14·40	13·24
Ash	4·44	10·00	2·32	12·10	12·11	30·53
Fixed carbon	36·68	36·43	46·28	39·40	72·66	54·92
Total	100·00	100·00	100·00	100·00	100·00	100·00
Sp. Gr.	1·10	1·33	1·27	1·35	1·40	1·75

Table II

	I Palana Lignite.	II Makerwal Coal.	III Mach Coal.	IV Dandot Coal.	V Jammu Anthracitic Coal.	VI Jammu Impure Coal
Total sulphur	1·10	4·61	3·95	9·08	1·94	1·00
Fixed sulphur	0·62	1·55	1·74	4·77	1·08	0·61
Volatile S.	0·48	3·06	2·21	4·31	0·86	0·39
Non-Combustible	0·61	0·08	0·18	0·17	0·03	0·03
Combustible S.	0·49	4·53	3·77	8·91	1·91	0·97
Pyritic S.	0·15	1·15	0·42	5·46	1·02	0·04
Sulphate S.	0·06	0·03	0·26	1·60	0·11	0·14
Organic S.	0·88	3·41	3·15	1·98	0·81	0·80

Explanation of Table II

The values of different sulphur forms are given in percentages of coal. Total sulphur in coal is determined separately. Fixed sulphur is the amount of sulphur in coke calculated in terms of coal. The fixed sulphur together with volatile sulphur forms the total sulphur. Non-combustible sulphur is the amount of sulphur in ash calculated in terms of coal. The non-combustible sulphur and the combustible sulphur together form the total sulphur. Pyritic sulphur, sulphate sulphur and the organic sulphur, determined separately, all go to make the total sulphur.

Anomalies in the Wing-structure of *Pompilus wroughtoni* Cam. (Hymenoptera).

By

A. C. Sen, M.Sc.

The chief characteristic features by which the members of the genus *Pompilus* are distinguished from one another is the structure of the wings. The wings are generally well developed, the fore-wings with one radial and two or three complete cubital cells are usually found. In some the wings are partially or entirely hyaline, whereas in others they are dark-fuscous or ferruginous-yellow.

In *Pompilus wroughtoni*, which are quite small insects measuring about 17 mm., the wings are entirely sub-hyaline. The most striking feature I have come across is not that of variation in the colouration but a distinct difference in the wing venations of the fore-wing. It must be remembered that the wings are constant in members of the same species.

Cameron writing about this in the *Mem. Manch. Lit. Ph. Soc.* (4), IV; 1891, page 464, remarked about the wing venations as " the second cubital cellule is much longer at the bottom compared with the third, the third being of the length of the space bounded by the first transverse cubital and the first recurrent, the latter being received at a greater distance from the transverse cubital, the second recurrent is received at the apical fourth of the cellule not before the middle." The figure as shown by Cameron is reproduced in the text figure B. In his specimen, Cameron found the costal vein running parallel and nearer the anterior margin of the wing. But in a specimen in the collection of the Indian Museum the venation is remarkably different.

The radial cell (Fig. A)* is distinctly divided by a small vein, joining it with the apical margin near the apex of the wing, and the second cubital cell is bounded by the first transverse cubital nervure and it also receives both the first and second recurrent nervures, the

fourth cubital cell does not receive the recurrent nervures. The costal vein runs considerably at a lower distance and is not parallel to the anterior margin. These differences can be readily understood by comparing the text-figures A and B.

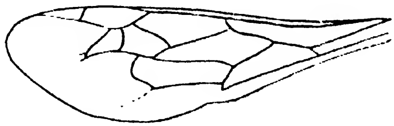

fig. A

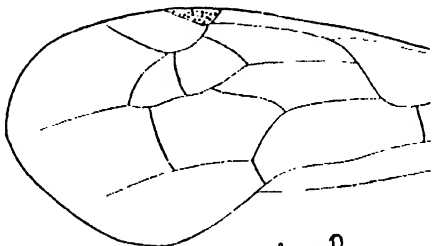

fig. B

I may also add that the habitat as mentioned in the *Fauna of British India*, Hymenoptera, Vol. I, is Poona in the Bombay Presidency, whereas this specimen was collected at Barrackpore, in Bengal. Cameron did not mention whether his specimen was a male or female. The specimen under report is a female and it possesses a long ovipositor at the anal extremity.

I express my sincere thanks to Dr. H. S. Pruthi, Officer-in-Charge of the Entomological Section of the Zoological Survey of India for allowing me all the facilities for working in his laboratory.

Notes on the Diphenyl Crystals

BY

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In course of a study of the diphenyl crystals, the writer of this note was struck by the unusual value of the axial ratio $\frac{c}{b}$ in the morphological descriptions in Groth's Chemische Krystallographie.[1] It is stated therein that—

$a:b:c = 1{\cdot}4428 : 1 : 5{\cdot}4331$ and $\beta = 94°46'$ (Mieleitner).

Forms......$c(001)$, $m(110)$, $r(101)$, $b(010)$ and $a(100)$.

Crystals thin and tabular parallel to $c(100)$, twins on $(10\bar{4})$, axial plane parallel to $b(010)$.

By X'ray methods, Hengstenberg and Mark [2] have given the following data for the diphenyl crystals—

$$a = 8{\cdot}22\text{Å},\quad b = 5{\cdot}69\text{Å},\quad c = 9{\cdot}50\text{Å},\quad \beta = 94°8'$$

Axial ratio $a:b:c = 1{\cdot}445 : 1 : 1{\cdot}670$

They have therefore remarked that the value of $\frac{c}{b}$ as given in Groth's treatise is incorrect. The forms observed by them were $c(001)$, $m(110)$, $a(100)$ and an ortho-dome $(\bar{2}01)$.

The diphenyl crystals obtained by the author from an alcoholic solution were also thin and tabular parallel to $c(001)$, with the forms $c(001)$, $m(110)$ and an ortho-dome $(\bar{h}0l)$ and there was also a large number of twins on this ortho-dome. The angular values as measured

[1] Groth—Chemische Krystallographie, Vol. 5 (1919), p. 7.

[2] Hengstenberg and Mark—Zs. f. Kryst., Vol. LXX (1929), pp. 285-87.

in a Fuess goniometer are given below, and Mieleitner's [1] data are also stated for comparison:

Biswas		Mieleitner
$001 \wedge \bar{1}10 - 93°10'$	cm	$001 \wedge 110 - 87°17'$
$110 \wedge \bar{1}10 - 70°14'$	mm	$110 \wedge 1\bar{1}0 - 110°22'$
$001 \wedge \bar{h}ol - 70°45'$	cr	$001 \wedge 101 - 70°37\frac{1}{2}'$
$\bar{1}10 \wedge \bar{h}ol - 58°46'$	mr	$110 \wedge 101 - 56°30'$
$001 \wedge ho\bar{l} - 109°21'$		

It appears from the above figures that the ortho-dome r is (101) according to Mieleitner and ($\bar{h}$ol) according to the author. The angular value $70°45'$ ($70°37\frac{1}{2}'$ of Mieleitner) is that of the angle between 001 and the ortho-dome lying on the negative side of the clino-axis a. The angle between 001 and the ortho-dome lying on the positive side of the axis a is $109°\ 21'$ and not $70°37\frac{1}{2}'$ as stated by Mieleitner. The form r is therefore ($\bar{h}$ol) and not (101). This is also corroborated by the fact that by calculation $rm = 56°26'$ when rm is $101 \wedge 110$ and $58°36'$ when rm is $\bar{h}ol \wedge \bar{1}10$. By measurements the angle rm has been found to be $58°46'$, showing that the face r is $\bar{h}$ol.

It has already been mentioned that Hengstenberg and Mark have recorded the occurrence of an ortho-dome $\bar{2}01$. If $\bar{h}$ol be substituted by $\bar{2}01$ it is found that the axial ratio $\frac{c}{b}$ becomes 1·670 (by using Mieleitner's value for β) and this is quite in agreement with the ratio given by the X'ray work. The form r is therefore ($\bar{2}01$).

Calderon [2] has observed a twin on (101) with a twinning angle $c \wedge c'$ $(001 \wedge 00\bar{1}) = 37°5'$. This 101 face is really $\bar{2}01$ as established above. The composition plane of the twin is therefore $\bar{2}01$. The twins obtained by the author are also on $\bar{2}01$ and the angle $c \wedge c' = 37°20'$. In Groth's treatise it has been stated that Calderon has observed a twin on 104 but in the original article of Calderon [2] the author has failed to find any mention of such a twin.

My best thanks are due to Sir C. Venkata Raman who initiated me to the study of the diphenyl crystals and to Mr. K. L. Narasinham of the Benares Hindu University who kindly prepared for me several crops of these crystals.

[1] Mieleitner—*Zs. f. Kryst.*, Vol. LV (1915-20), pp. 51-53.
[2] Calderon— „ „ „ IV (1880), p. 240.

The Systematic Value of Leaf Ash *

BY

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In the earlier period, the classification of plants was based exclusively on the study of morphological characters. But with the progress of science in general, anatomy, chemistry and other internal characters of plants have been gradually worked out; this has led to the establishment of different schools of systematists, who have based their classifications on the basis of one character or another.

The first attempt at the anatomical grouping of plants may be ascribed to Weddel.[1] In his monograph on the *Urticaceae*, he has made use of cystoliths of the leaf, in the determination of the genera. The real founder of the anatomical method was Professor Rodlkofer.[2] His investigations have brought to light the fact, that the anatomical method was not only important but sometimes a necessity in Systematic Botany. In Paleo-Botany, the nature of fossil specimens is determined mostly by their anatomy. Bertrand's [3] work on the anatomy of *Gnetum* and *Conifers*, A. Engler's [4] determination of the affinity of *Rutaceae Simarubaceae* and *Bursceraceae*, Bureau's [5] systematic determination of American *Bignonia* and M. Fernand's [6] work on the leaf anatomy of *Celtis*, *Ulmus*, etc., are main contributions to Systematic Botany, from the anatomical point of view.

The chemical properties of plants, can be made use of in the study of Systematic Botany. Mucilage for instance is characteristic of *Malvaceae*, resin, of *Conifers* and strychnine, of *Strychnos*. Even in purely morphological grouping of plants, chemical characters, such as starchy or oily nature of endosperm, have often been referred to, in assigning the systematic position to certain plants. A stable system of classification is only possible, when not only morphological characters, but anatomical, chemical and other characters of plants are as a whole taken into consideration.

* Read before the Indian Science Congress held at Allahabad in January, 1930.

A new method has been proposed by Professor H. Molisch;[7] although it cannot be extended to the differentiation of all genera and species, yet its importance lies in the conclusive determination of some plant families by a mere trace of leaf ash. The method is based on the fact, that plants absorb from the soil a large quantity of minerals in solution, and these minerals are transported to the leaves where various complex substances are manufactured. There is a gradual partial deposition of these minerals on the cell-walls as well as within the cells. The steady increase in the contents of the leaves is apparent from the table that has been worked out by J. M. Coulter[8] which is given below.

Fagus sulvastica.	May.	June.	July.	August.	Sept.	Oct.	Nov.
Percentage of dry matter	23·35	40·21	43·6 1	50·74	47·42	40 37	45·56
Percentage of ash	4·67	5·20	7·15	9·03	8·90	10·80	11·44

Rubinia pseudoacacia.	May.	July.	Sept.	Oct.
Percentage of dry matter	26·50	35·90	44·30	44·60
Percentage of ash	6 25	7·75	8·22	11·74

The mode of deposition of the minerals on cell-walls and within the cells, and their chemical nature, are characteristics of certain group of plants. These characteristics may be conveniently used in the systematic determination of some families of plants.

In practice, a small piece of a leaf about an inch square, is heated on a piece of platinum foil over the flame of a spirit-lamp for 2 to 3 minutes until the leaf is reduced to ashes. This is then carefully removed to a glass slide and mounted with a drop of aniline oil. The aniline oil removes quickly all air bubbles and a clear view of the anatomical structure of the leaf and its contents may then be obtained under a microscope. A piece of mica may be used instead of a platinum foil, but then it requires to be heated by a stronger flame; further the mica foil soon gets out of use. The ash may otherwise be prepared by simply burning the leaf over a flame. The epidermal ash of the stem in many instances has characteristics identical with those of the leaf ash. Both young and old leaves have almost identical ash contents, only in the latter contents are larger and more numerous.

A large number of plant families have been investigated according to this method; a general survey of some of the families is given

here. In all cases attempts have been made to examine the maximum number of the genera available, which include mostly plants of Bengal and Assam, and in some cases those of Burma, Europe and America.

A short key to the following families is given below :—

(A) Cystoliths usually present :

(*a*) Cystoliths usually elongated with serrate margin ... ... *Acanthaceae.*

(*b*) Cystoliths usually spherical, sometimes associated with trichomes ... ... *Urticaceae.*

(B) Cystoliths absent :

(X) Raphide bundles and rectangular crystals present.

(*x*) Raphide bundles large, distribution more or less parallel, rectangular crystals also large sometimes twin ... ... *Pandanaceae.*

(*y*) Raphide bundles small, distribution irregular, spherical and stellate bodies also present ; twin crystals none .. ... *Vitaceae.*

(Y) Raphide bundles and rectangular crystals absent.

(*) Bodies of irregular shape, of amorphous mass of calcium oxalate ... ... *Solanaceae.*

(**) Bodies of regular shape of various composition.

(@) Numerous bodies of various shapes, some almost square with slight lateral depression in the middle, others spherical at top view but conical at lateral view, guard cells sometimes distinct even in ash ... ... *Gramineae.*

(@@) Numerous small almost square deckplate cells on surface view more or less rectangular with serrate margin and a central circular depression but on lateral view more or less conical with truncate apex and serrate base ... *Musaceae.*

Acanthaceae.

It is a large family comprising 240 genera with about 2,000 species, of which 74 genera have been investigated. The general characters of most of the genera are :—

1. The wide distribution of the cystoliths.

2. Oxalate of lime in form of prismatic or acicular crystals both in the epidermis and mesophyll of the leaf.

Following genera are however conspicuous by absence of the cystoliths :—

1. *Nelsonia.*	3. *Acanthus.*
2. *Thunbergia.*	4. *Aphelandra.*

Cystoliths as occur in this family may be of the following types :—

(*a*) Cystoliths elongated, pointed at both ends, margin serrate. Fig. 1.*b*.

Rostellaria, *Sanchezia.*

(*b*) Cystoliths elongated, pointed at one end obtuse at the other, margin slightly serrate. This is the most general form prevalent in the family. Fig. 1. *a*.

Andrographis, Spirostigma, Stephanophysum, Echinacanthus, Chaetothylax.

(*c*) Cystoliths elongated but obtuse at both ends. Fig. 1. *c*.

Extreme modification of this type is sometimes represented by more or less spherical forms with projections of a spiny nature over the surface. Fig. 1. *d*.

Justicia, *Adhatoda,* *Barleriola.*

(*d*) Cystoliths of the type (*b*) in pairs joined by their obtuse ends. Fig. 1. *e*.

Urticaceae.

It is an important tropical and subtropical family containing a large number of genera and species. They are distinguished mostly by the presence of—

(*) cystoliths, which are of the following types:—

(a) Spherical bodies with superficial protuberances as in some Ficus. Fig. 6. b.

(b) Fusiform as in Pilia.

(c) Oblong as in Pellionia.

(**) Numerous hairs or trichomes which are mostly curved at the apex. Fig. 6. a.

U. dioeca, *Forskohela*, *Ficus hispida*.

Hairs and trichomes are sometimes absent in some species. They are composed mostlyof calcium carbonate and silica. Trichomes and hairs may be free or may be associated sometimes with spherical cystoliths at the base. Fig. 6. c.

Pandanaceae.

It is a small family, consisting of three genera, *Saranga*, *Freycintia*, *Pandanus*. *Pandanus* alone was available for examination. It is, however, distinguished by the presence of

(a) Numerous rectangular crystals; some of the crystals are very small but are associated with a number of larger crystals which are often twin, measuring an angle of 140° as in *Pandanas minor*. Fig. 3. a.

(b) Numerous large raphide bundles distributed at regular intervals parallel to the long axis of the leaf. Fig. 3. b.

Raphide bundles are also present in *Vitaceae* but *Pandanaceae* is distinguished from *Vitaceae* by

1. The magnitude of the raphide bundles which is about 2-3 times as large as that of Vitaceae.

2. The parallel distribution of the raphide bundles.

3. The occurrence of twin crystals.

4. The absence of numerous spherical bodies of calcium oxalate.

Vitaceae.

It is a small family, mostly tropical and subtropical, containing 11 genera with 450 species. Bengal however is represented by 2 genera, *Vitis* and *Leea* with a total of 34 species. Common characters by which this family is distinguished are :—

1. Numerous small bunch of raphide needles distributed irregularly. Fig. 4.*a*.

2. Numerous regular spherical bodies with star-shaped outline known as clustered crystals composed of calcium oxalate arranged in row. Fig. 4.*b*.

Vitis quadrangularis, V. aestivalis, Leea marcophylla.

Solanaceae.

The family comprises about 72 genera with 1,500 species, of which 20 genera have been investigated. The distinguishing characters of the family are the occurrence of—

1. Numerous amorphous masses of calcium oxalate closely aggregated into numerous irregular bodies of no definite shape. Fig. 5.*b*.

The calcium oxalate as occurs in this family may be loosely packed into groups known as

(*) Crystal sand as in

Datura, Capsicum, Nicotina, Solanum; or it may occur

(**) in the form of crystals of primatic, cubic or octohedral shape. Fig. 5.*a*.

Iochroma, solocha, and some *Atropa.*

Graminaceae.

Most plants of this family are distinguished by :—

1. Great deposition of silica on the cell wall; epidermal cells with serrate lateral wall arranged in parallel rows.

2. Interposed at regular intervals are numerous almost square bodies with slight lateral-depression. These bodies are mostly of calcium carbonate associated with certain amount of silica. Fig. 7.*b*.

3. Numerous small bodies, more or less spherical on top view but conical at lateral view, distributed regularly or irregularly. Fig. 7 *a*.

4. Sometimes certain amount of silica and calcium carbonate are deposited on guard cells, so that the characteristic stoma of the family is retained even in the leaf ash.

Musaceae.

Of the six genera of this family only Ravenala and Musa have been examined with reference to their leaf ash. They are characterised by the presence of numerous small, almost square, deck plate cells, closely packed, arranged in rows along the veins. Margin of these cells are serrate; on surface view they appear more or less square with serrate margin and a central circular depression in the middle but on lateral view they appear more or less as conical structures with truncate apex and a serrate base. Fig. 8.

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Literature.

[1] Weddel. Monogr. des. Urticaceae.

[2] Rodlkofer, L. Uber. d. Methoden. in d. Bot. Syst. insbes d anat. Methode. Glied d. Sapindae.

[3] Bertrand, M. Ann. Sci. nat. ser. 5.t. xx. 1874. Des characteres que l'anatpent fourmir a la classificat.

[4] Engler, A. Natural Pflanzenfam.

[5] Bureau, Characters tires de la struct de la de la la tige pour la classif des B.

[6] Fernand, M. Stricture. comparee des femillies de memes age et de dimansions differentis in compt. rendu., Vol. 173, pp. 1113-16.

[7] Molisch, H. Microchemie der Pflanze.

[8] Coulter, J. M. Text-book of Botany, Vol. I, p. 354.

Explanation of Figures.

Fig.
1. *a*. Cystoliths pointed at one end. x. 65.
   b. Cystolith pointed at both ends. x. 65.
   c. Obtuse cystolith. x. 75.
   d. Spherical cystolith. x. 60.
   e. Double cystolith. 65.
3. *a*. Twin crystals. x. 175.
   b. Raphide bundle. x. 100.
4. *a*. Raphide bundle. x. 100.
   b. Spherico-stellar body. x. 125.

Fig.
5. *a*. Crystal sand. x. 200.
   b. Irregular bodies. x. 110.
6. *a*. Trichome. x. 60.
   b. Spherical cystolith. x. 65.
   c. Trichome with spherical cysto'ith. x. 60
7. *a*. Pyramidal bodies. x. 125.
   b. Crystaline structure. x 175.
8. Deck Plate Cells. x 200.

On a Set of Self-repeating Lines cutting the Sides of Polygons.

BY

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1. The following proposition has been recently established by me in the Bulletin of the Calcutta Mathematical Society [1] :

"If from a point P on the side AB of the $\triangle$ ABC we draw a line PQ making an angle θ with BA and cutting AC at Q, and if QR and RS be drawn in the same way making angles ϕ and ψ with AC and CB, then we shall come back to the starting point P after a single repetition if θ, ϕ and ψ satisfy the relation

$$\text{Sin } (A+\theta) \text{ Sin } (B+\psi) \text{ Sin } (C+\phi) = \text{Sin } \theta \text{ Sin } \phi \text{ Sin } \psi.\text{'}$$

I have also shown that when

$$\theta = \frac{\pi - A}{2}, \quad \phi = \frac{\pi - C}{2}, \quad \psi = \frac{\pi - B}{2},$$

PQ becomes the chord of contact of a circle touching AB and AC, QR the chord of contact of a circle touching AC and BC and RS the chord of contact of a circle touching BC and AB. For the sake of shortness PQ, QR and RS will in such a case be called *cyclic* lines.

[1] Bulletin, Calcutta Mathematical Society, December, 1928, No. 4, Vol. XIX.

When $\theta=B$, $\phi=A$, $\psi=C$,

PQ, QR and RS become parallel to CB, BA and AC.

When $\theta=C$, $\phi=B$, $\psi=A$,

PQ, QR and RS become antiparallel to CB, BA and AC.

In the present paper I shall generalise these propositions in the case of polygons of even and odd number of sides.

2. If we go on drawing cyclic lines from a point on a conic we do not get a system of six lines as in the case of a triangle but we simply get a rectangle inscribed in the conic. This at once follows from the fact that the chord of contact of a circle touching a conic at two points is always parallel to one of the two axes.

It can be shown by easy geometric considerations that in the cases of a square or a rhombus also we get inscribed rectangles.

Let us now take up the case of a general quadrilateral ABCD.

Let the lengths of the sides DA, AB, BC and CD be a, b, c and d.

Let P, Q, R and S be points taken on the sides DA, AB, BC and CD.

First let PQ, QR, etc., be cyclic lines as defined before and let the distance $AP=x$.

$$AP=AQ=x.$$

$$\because BP=BR=b-x.$$

$$\because CR=CS=c-(b-x)=c-b+x.$$

$$\therefore DS=DT=b+d-c-x.$$

$$\text{But } DP=a-x.$$

Hence if T be the next position of P, T cannot coincide with P unless

$$a=b+d-c$$

$$\text{or } c+a=b+d$$

i.e., the sum of the opposite sides must be equal. If this be not the case we have $PT=b+d-c-a$,

$\therefore$ if we go on drawing lines in this way we shall not come back to the starting point as in the case of a triangle but we should get a series of points on any one of the sides such that the distance between any two consecutive points on the same side remains constant.

3. Though cyclic lines do not repeat themselves in the case of a quadrilateral they do so in the case of a pentagon This can be established as follows. The construction being the same as that in the case of a quadrilateral we need introduce only a fifth side of length e and take U to be the second position of P on AB.

Let $BP=x$.

$BQ=BP=x \quad \therefore \quad CQ=CR=b-x.$

$\therefore DR=DS=c-b+x$ and $ES=ET=b+d-c-x.$

$\therefore AT=AU=e+c-b-d+x.$

$\therefore BU=a+b+d-e-c-x.$

Now the point U is on the same footing as P.

$\therefore$ if by performing a similar construction we get a point U′, we must have

$$BU'=(a+b+d-c-e)-(a+b+d-c-e-x)=x.$$

$\therefore$ U′ will coincide with P.

It is at once evident that the distance BU will always be of the form $k \pm x$ where k will be a function of the lengths of the sides.

In the case of even-sided polygons BU will be of the form $k+x$ and in the case of odd-sided polygons it will be of the form $k-x$.

$\therefore$ in the case of even-sided figures BU′ will be

$$k+(k+x)=2k+x$$

and in the case of odd-sided figures BU′ will be

$$k-(k-x)=x.$$

Hence we have the following generalisation :—

"If starting from a point P of a rectilinear figure, we draw straight lines PQ, QR, RS, etc., the chords of contact of circles touching two consecutive sides, we always get a closed figure in the case of odd-sided polygons and an unclosed one in the case of even-sided polygons. In the case of odd-sided polygons we come back to the starting point after twice traversing the sides, whereas in the case of even-sided polygons we get a range of points on every side such that the distance between two consecutive points on the same side remains constant, but if the even-sided polygon be such that the sum of the even sides is equal to the sum of the odd sides then we get a closed figure after traversing the sides only once."

4. We shall now take up the case corresponding to lines drawn parallel to the sides of a triangle. In the case of a quadrilateral ABCD, PQ will be parallel to DB, QR parallel to AC, etc.

From the property of parallels and similar triangles we at once have in the case of a quadrilateral

$$\frac{DP}{AP}=\frac{BQ}{QA}=\frac{BR}{RS}=\frac{DS}{CS}$$

$\therefore$ SP$\|$AC.

$\therefore$ we come back to the starting point after traversing the figure only once.

In the case of a pentagon ABCDE, PQ$\|$EB, QR$\|$AC, RS$\|$BD, ST$\|$CE and TU$\|$DA.

EB will be called the diagonal corresponding to PQ, AC the diagonal corresponding to QR and so on.

This nomenclature will be adopted in the case of a hexagon as also in the case when PQ, QR, etc., are antiparallel to EB, AC, etc.

Let the sides EA, AB, BC, CD and DE of a pentagon be a, b, c, d and e units of length and let AP$=x$.

Let PQ be parallel to the corresponding diagonal EB, QR parallel to AC and so on.

Making use of the property of similar triangles we have

$$AQ=\frac{b}{a}x$$

$$BQ=\frac{b(a-x)}{a}$$

$$BR=\frac{c}{a}(a-x)$$

$$CR=\frac{cx}{a}$$

$$CS=\frac{dx}{a}$$

$$DS=\frac{d(a-x)}{a}$$

$$DT=\frac{e(a-x)}{a}$$

$$ET=\frac{ex}{a}$$

$$EU=x \quad \therefore \; AU=a-x.$$

Now the point U is on the same footing as P.

$\therefore$ the distance of the next point thus obtained (from A) will be

$$a-(a-x)=x.$$

From the method of proof it is evident that the theorem holds good in the case of all odd-sided figures.

In the case of a hexagon let V be the next position of P and let the length of the sides be FA, AB,...EF be a, b, c, d, e, f.

As before let $AP=x$.

From similar triangles

$$\frac{x}{a}=\frac{AQ}{b} \quad \therefore \; AQ=\frac{b}{a}x \text{ and } BQ=\frac{b(a-x)}{a}.$$

$$\frac{CR}{c}=\frac{AQ}{b}=\frac{x}{a} \quad \therefore \; CR=\frac{cx}{a} \text{ and } BR=\frac{c(a-x)}{a}$$

$$\frac{CR}{c}=\frac{CS}{d} \quad \therefore \; CS=\frac{d}{c}CR=\frac{dx}{a} \text{ and } DS=\frac{d(a-x)}{a}.$$

$$\frac{ET}{e}=\frac{CS}{d} \quad \therefore \; ET=\frac{e}{d}CS=\frac{ex}{a} \text{ and } TD=\frac{e(a-x)}{a}.$$

$$\frac{EU}{f}=\frac{ET}{e}=\frac{x}{a} \quad \therefore \; EU=\frac{fx}{a} \text{ and } FU=\frac{f(a-x)}{a}$$

$$\frac{AV}{a}=\frac{EU}{f}=\frac{x}{a} \quad \therefore \; AV=x.$$

$\therefore$ V coincides with P.

It is evident that the above proof applies in the general case. Hence we get the following proposition :—

"If from a point on one side of a polygon we go on drawing lines parallel to the corresponding diagonals, we shall come back to the starting point after traversing the figure once or twice according as the figure is even or odd-sided."

5. In the case of a quadrilateral ABCD let PQ be antiparallel to the corresponding diagonal DB.

Then in the previous notation

$$AQ=\frac{b}{a}x \;\ldots\; DS=d-\frac{c}{d}\left\{c-\frac{b}{c}\left(b-\frac{ax}{b}\right)\right\}$$

$$\frac{DS}{DT}=\frac{a}{d} \quad \therefore \; DT=\frac{d}{a}\left[d-\frac{c}{d}\left\{c-\frac{b}{c}\left(b-\frac{ax}{b}\right)\right\}\right]$$

$$\therefore\ AT = a - \frac{d}{a}\left[d - \frac{c}{d}\left\{c - \frac{b}{c}\left(b - \frac{ax}{b}\right)\right\}\right] = P - x \text{ say}$$

$\therefore$ if P′ be the next position of P,

$$AP' = P - (P - x) = x.$$

Performing the above construction in the case of a pentagon we have

$$AQ = \frac{ax}{b},\quad BQ = b - \frac{ax}{b}.$$

$$\frac{BQ}{BR} = \frac{c}{b}\quad \therefore\ BR = \frac{b}{c}\,BQ = \frac{b}{c}\left(b - \frac{ax}{b}\right).$$

$$\text{and } CR = c - \frac{b}{c}\left(b - \frac{ax}{b}\right).$$

$$\frac{CR}{CS} = \frac{d}{c}\quad \therefore\ CS = \frac{c}{d}\left\{c - \frac{b}{c}\left(b - \frac{ax}{b}\right)\right\}$$

$$\text{and } DS = d - \frac{c}{d}\left\{c - \frac{b}{c}\left(b - \frac{ax}{b}\right)\right\}.$$

$$\frac{DS}{DT} = \frac{e}{d}\quad \therefore\ DT = \frac{d}{e}\left[d - \frac{c}{d}\left\{c - \frac{b}{c}\left(b - \frac{ax}{b}\right)\right\}\right].$$

$$\therefore\ TE = e - \frac{d}{e}\left[d - \frac{c}{d}\left\{c - \frac{b}{c}\left(b - \frac{ax}{b}\right)\right\}\right].$$

$$\frac{TE}{EU} = \frac{a}{e}$$

$$\therefore\ EU = \frac{e}{a}\left[e - \frac{d}{e}\left\{d - \frac{c}{d}\left(c - \frac{b}{c}\,b - \frac{ax}{b}\right)\right\}\right]$$

$$\therefore\ AU = a - \frac{e}{a}\left[e - \frac{d}{e}\left\{d - \frac{c}{d}\left(c - \frac{b}{c}\,b - \frac{ax}{b}\right)\right\}\right].$$

Thus AU is of the form $P-x$ where P is a function of the sides.

If U′ be the next position of U we must have

$$AU' = P-(P-x) = x.$$

$\therefore$ U′ coincides with P.

It is also evident that this will hold good in the case of all even-sided polygons.

Hence we have the proposition :—

"If from a point on one side of a polygon we go on drawing antiparallels to the corresponding diagonals we always come back to the starting point after twice traversing the sides."

On the Locus traced out by two Homographic Pencils

BY

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The object of the present paper is to give simple proofs of some of the anharmonic properties of conics and to investigate the nature of the locus of the points of intersection of the corresponding rays of two pencils in homography. Though the propositions on anharmonic properties are not new, the proofs given are all original and simpler than those usually found in text-books.

Proposition I.

The locus of a point such that the anharmonic ratio of the pencil formed by joining it to four fixed points is constant is a conic.

Let P (x_1, y_1, z_1), Q (x_2, y_2, z_2), R (x_3, y_3, z_3), S (x_4, y_4, z_4) be the four fixed points and O (ξ, η, ζ) the variable point.

We have to find the locus of O when

$$O(PQRS) \text{ is const.} = \lambda \text{ say.}$$

The equation of OP is

$$\begin{vmatrix} x, & y, & z \\ \xi, & \eta, & \zeta \\ x_1, & y_1, & z_1 \end{vmatrix} = 0$$

i. e., $$x(\eta z_1 - \zeta y_1) - y(\xi x_1 - \zeta z_1) + z(\xi y_1 - \eta x_1) = 0.$$

If ABC be the triangle of reference and OP cuts AB at P′, putting $z = 0$ in the above equation CP′ is found to be

$$x = \frac{\xi z_1 - \zeta x_1}{\eta z_1 - \zeta y_1}\, y.$$

If OQ, OR, and OS cut AB at Q′, R′, and S′ the equations of CQ′, CR′ and CS′ are similarly found to be

$$x=\frac{\xi z_2-\zeta x_2}{\eta z_2-\zeta y_2}y,$$

$$x=\frac{\xi z_3-\zeta x_3}{\eta z_3-\zeta y_3}y$$

and

$$x=\frac{\xi z_4-\zeta x_4}{\eta z_4-\zeta y_4}y \text{ respectively.}$$

Now by our construction

$$O\,(PQRS)=C(P'Q'R'S')$$

If then CP′, CQ′, CR′, and CS′ be denoted by $x=m_1y$, $x=m_2y$, $x=m_3y$ and $x=m_4y$ respectively, we have

$$m_1-m_2$$

$$=\frac{\xi z_1-\eta x_1}{\eta z_1-\zeta y_1}-\frac{\xi z_2-\zeta x_2}{\eta z_2-\zeta y_2}=\begin{vmatrix}\xi, & \eta, & \zeta\\ x_1, & y_1, & z_1\\ x_2, & y_2, & z_2\end{vmatrix}\div\{(\eta z_1-\zeta y_1)(\eta z_2-\zeta y_2).\}$$

Finding the values of m_3-m_4, etc., in the same way we get

$$C(P'Q'R'S')=\frac{(m_1-m_2)\,(m_3-m_4)}{(m_1-m_4)\,(m_3-m_2)}$$

$$=\frac{(\xi y_1z_2)\,(\xi y_3z_4)}{(\xi y_1z_4)\,(\xi y_3z_2)}$$

$$=\lambda,$$

where λ is the constant anharmonic ratio.

Hence putting x, y, z for ξ, η, ζ, we get the locus of O in the elegant determinant form

$$\frac{\begin{vmatrix}x, & y, & z\\ x_1, & y_1, & z_1\\ x_2, & y_2, & z_2\end{vmatrix}\times\begin{vmatrix}x, & y, & z\\ x_3, & y_3, & z_3\\ x_4, & y_4, & z_4\end{vmatrix}}{\begin{vmatrix}x, & y, & z\\ x_1, & y_1, & z_1\\ x_4, & y_4, & z_4\end{vmatrix}\times\begin{vmatrix}x, & y, & z\\ x_3, & y_3, & z_3\\ x_2, & y_2, & z_2\end{vmatrix}}=\lambda,$$

Corollary.

The locus is at once seen to be a conic and as the above equation is identically satisfied by putting $x=x_r$, $y=y_r$, $z=z_r$, $(r=1, 2, 3, 4,)$ the conic passes through the four fixed points.

It has been assumed that no three of the points P, Q, R, S lie on a straight line.

Proposition II.

The anharmonic ratio of the pencil formed by joining four fixed points on a conic to a variable fifth point on it is constant.

This theorem is easily established by taking the conic in the form

$$xy=z^2.$$

The co-ordinates of any point on the conic take the parametric form

$$x : y : z=t^2 : 1 : t.$$

Taking $(t_r^2, 1, t_r)$, $(r=1, 2, 3, 4)$ to be the four fixed points and $(t^2, 1, t)$ as the variable point the proposition follows at once.

Proposition III.

The locus of the points of intersection of the corresponding rays of two homographic pencils is a conic.

Let the vertices of the pencils be taken as the vertices B and C of the triangle of reference.

Any line of the first pencil is $z=kx$.

Let the corresponding line of the second pencil be $x=k'y$, the relation between k and k' being

$$k' = \frac{Pk+Q}{Rk+S}$$

Eliminating k and k' between these three equations we get the equation of the locus

$$\frac{x}{y}=\frac{P\frac{z}{x}+Q}{R\frac{z}{x}+S}$$

i.e., $Sx^2-Pyz+Rzx-Qxy=0$

The locus evidently passes through the vertices of the two pencils.

I shall now state the reciprocal theorems without giving complete proofs which can be at once obtained by similar methods.

Reciprocal of Proposition I.

The envelope of a line forming a range of constant anharmonic ratio λ with four fixed straight lines

$$l_r x + m_r y + n_r z = 0$$

is the conic.

$$\frac{\begin{vmatrix} l, & m, & n \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix} \times \begin{vmatrix} l, & m, & n \\ l_3, & m_3, & n_3 \\ l_4, & m_4, & n_4 \end{vmatrix}}{\begin{vmatrix} l, & m, & n \\ l_1, & m_1, & n_1 \\ l_4, & m_4, & n_4 \end{vmatrix} \times \begin{vmatrix} l, & m, & n \\ l_3, & m_3, & n_3 \\ l_2, & m_2, & n_2 \end{vmatrix}} = \lambda,$$

which evidently touches the four fixed lines.

Reciprocal of Proposition II.

The anharmonic ratio of the range formed by four fixed tangents of a conic with a variable fifth tangent is constant.

The proof follows easily by taking any point on the conic to be

$$t^2 : 1 : t \quad \text{as before.}$$

Reciprocal of Proposition III.

The straight lines joining the corresponding points of two homographic ranges envelopes a conic touching the bases of the two ranges.

If the two bases be taken as the two Cartesian axes, the equation of the envelope is

$$(Px - Sy + Q)^2 + 4(SP - QR)xy = 0,$$

where P, Q, R, S are the constants in the relation giving the homography as in Proposition III.

I shall now investigate the nature of the locus obtained in Proposition III.

The equation of the conic being

$$Sx^2 - Pyz + Rzx - Qxy = 0$$

if the co-ordinates be Areal, this will be an hyperbola, parabola or ellipse, according as

$$(P+Q+R)^2-4P(S+Q)\lesseqgtr 0 \qquad \ldots \qquad \ldots \quad (1)$$

It will be a rectangular hyperbola, if

$$\frac{P\cos A}{a}+\frac{S-R}{b}\cos B+\frac{S+Q}{c}\cos C=0 \qquad \ldots \quad (2)$$

It will be a circle, if

$$\frac{P}{a^2}=\frac{S-R}{b^2}=\frac{S+Q}{c^2} \qquad \ldots \quad (3)$$

It will be a pair of straight lines, if

$$\frac{P}{R}=\frac{Q}{S} \qquad \ldots \qquad \ldots \qquad \ldots \quad (4)$$

The geometric interpretations of (1), (2), (3) and (4) will now be found by the help of the following well-known proposition. The angle ω between the lines of which the trilinear equations are

$$l_1\alpha+m_1\beta+n_1\gamma=0$$
$$l_2\alpha+m_2\beta+n_2\gamma=0$$

is given by

$$\tan\omega=\frac{\begin{vmatrix} \operatorname{Sin} A, & \operatorname{Sin} B, & \operatorname{Sin} C \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}}{\Sigma l_1 l_2-\Sigma(m_1 n_2+m_2 n_1)\cos A} \qquad \ldots \qquad \ldots \quad (5)$$

When transferred to trilinears the lines

$$z=kx$$

and $\qquad x=k'y,\ i.e.,\ x=\dfrac{Pk+Q}{Rk+S}y$ become

$$ak\alpha-c\gamma=0$$

and $\qquad a\alpha-b\dfrac{Pk+Q}{Rk+S}\beta=0$

If these lines be parallel we must have

$$\begin{vmatrix} ak & 0, & 0 \\ a & -b\dfrac{Pk+Q}{Rk+S} & 0 \\ a & b & c \end{vmatrix}=C$$

or $\quad Pk^2+(P+Q+R)K+S+Q=O. \qquad$ (A)

This is a quadratic equation in k and therefore there are in general two values of k for which the corresponding lines are parallel.

Hence in two homographic pencils there are generally two rays parallel to their correspondents.

The reality or otherwise of these will depend on the value of k. From (A) we see that there will be two real values of k, only one, or none according as

$$(P+Q+R)^2-4P(S+Q) \gtreqless 0,$$

a result giving the geometric significance of (1).

Thus in two homographic pencils there are in general two lines parallel to their correspondents. If these lines be real and distinct, the locus of the points of intersection of corresponding rays will be a hyperbola, if they are coincident the locus will be a parabola and if they are imaginary the locus will be an ellipse.

If k_1 and k_2 be the two values of k, the lines of the first pencil are

$$(ak_1\alpha-c\gamma)(ak_2\alpha-c\gamma)=0$$

or from (A)

$$c^2\gamma^2+\frac{P+Q+R}{P}ca\gamma\alpha+\frac{S+Q}{P}a^2\alpha^2=0$$

If the conic be a circle we have

$$\frac{P}{a^2}=\frac{S-R}{b^2}=\frac{S+Q}{c^2}=\frac{Q+R}{c^2-b^2}=\frac{P+Q+R}{c^2+a^2-b^2}$$

$\therefore$ the above equation becomes

$$c^2\gamma^2+\frac{c^2+a^2-b^2}{a^2}ca\gamma\alpha+c^2\alpha^2=0$$

i.e., $\alpha^2+2\gamma\alpha\cos B+\gamma^2=0$

But these are two isotropic lines through B.

Hence if the rays which are parallel to their correspondents be isotropic rays, the locus will be a circle, a result which may be inferred from the condition for an ellipse.

By (5) the angle between

$$ak_1\alpha - c\gamma = 0$$

and $$ak_2\alpha - c\gamma = 0$$

is given by

$$\tan \omega = \frac{abc}{2R'} \cdot \frac{k_1 - k_2}{a^2k_1k_2 + c^2 + ac(k_1 + k_2) \cos B}$$

where R′ is the circum-radius, or,

$$\frac{4R'^2}{a^2b^2c^2} \tan^2\omega = \frac{(k_1 - k_2)^2}{[a^2k_1k_2 + c^2 + ac(k_1 + k_2) \cos B]^2}$$

$$\frac{(P+Q+R)^2 - 4P(S+Q)}{\left\{ \frac{P \cos A}{a} + \frac{S - R}{b} \cos B + \frac{S + Q}{c} \cos C \right\}}$$

If the locus be a rectangular hyperbola, the denominator in the expression for tan ω vanishes.

∴ For a rectangular hyperbola $\tan^2 \omega = \infty$ or $\omega = \frac{\pi}{2}$ and conversely.

∴ If the two lines which are parallel to their correspondents be perpendicular to each other, the locus is a rectangular hyperbola. It will be seen that the conditions for ellipse, parabola and hyperbola may also be deduced from the above value of tan ω.

The condition for a pair of straight lines gives

$k' =$ a constant, which is impossible.

Thus the relation of the locus to the nature of the pencils is fully investigated analytically.

Incidentally it may also be shown that if the locus be a circle, the corresponding rays intersect at a constant angle. The angle between the lines

$$ak\alpha - c\gamma = 0$$

and $$a\alpha - b\,\frac{Pk+Q}{Rk+S}\,\beta = 0$$

is given by

$$\frac{R'}{abc} \tan \omega = \frac{Pk^2 + (P+Q+R)k + S + Q}{Lk^2 + Mk + N}$$

where

$$L = a^2R + Pab \cos C,$$
$$M = a^2S - Pbc \cos A + Rac \cos B + Qab \cos C$$
$$N = Sac \cos B - Qbc \cos A.$$

If this angle be constant, *i.e.*, independent of k, we must have

$$\frac{a^2\mathrm{R}+\mathrm{P}ab\cos\mathrm{C}}{\mathrm{P}}=\frac{a^2\mathrm{S}-\mathrm{P}bc\cos\mathrm{A}+\mathrm{R}ac\cos\mathrm{B}+\mathrm{Q}ab\cos c}{\mathrm{P}+\mathrm{Q}+\mathrm{R}}$$

$$=\frac{\mathrm{S}ac\cos\mathrm{B}-\mathrm{Q}bc\cos\mathrm{A}}{\mathrm{S}+\mathrm{Q}}.$$

These may also be written in the form

$$\frac{a^2\mathrm{R}}{\mathrm{P}}+ab\cos\mathrm{C}=-bc\cos\mathrm{A}+\frac{c^2\mathrm{S}}{\mathrm{S}+\mathrm{Q}}=-ab\cos\mathrm{C}+\frac{b^2(\mathrm{P}-\mathrm{Q})}{\mathrm{S}-\mathrm{R}}.$$

Eliminating R we have

$$\frac{\mathrm{P}}{a^2}\left\{\frac{c^2\mathrm{S}}{\mathrm{S}+\mathrm{Q}}-b^2\right\}=\mathrm{S}-\frac{b^2(\mathrm{P}-\mathrm{Q})(\mathrm{S}+\mathrm{Q})}{a^2\mathrm{S}+(a^2-c^2)\mathrm{Q}}$$

or, $\{c^2\mathrm{P}-a^2(\mathrm{S}+\mathrm{Q})\}\{a^2\mathrm{S}^2+b^2\mathrm{Q}^2+(a^2+b^2-c^2)\mathrm{QS}\}=0$

If $a^2\mathrm{S}^2+b^2\mathrm{Q}^2+(a^2+b^2-c^2)\mathrm{QS}=0$, it is easily seen that $\frac{\mathrm{Q}}{\mathrm{S}}$ will be imaginary.

∴ We must have

$$c^2\mathrm{P}-a^2(\mathrm{S}+\mathrm{Q})=0$$

or $$\frac{\mathrm{P}}{a^2}=\frac{\mathrm{S}+\mathrm{Q}}{c^2}.$$

In the same way eliminating Q we get

$$\frac{\mathrm{P}}{a^2}=\frac{\mathrm{S}+\mathrm{Q}}{c^2}=\frac{\mathrm{S}-\mathrm{R}}{b^2}$$

which are the conditions for a circle.

The converse is also true and can be proved by tracing some of the steps backwards.

A Problem in False Position

By

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In a paper published by me in the Bulletin of the Calcutta Mathematical Society I have established a general proposition relating to a set of repeating points on the sides of a triangle. In another paper published by me in the Journal of the Department of Science of the Calcutta University some of the results obtained in the first paper have been extended in the cases of even and odd-sided polygons. In the present paper I shall give a simple geometric construction for drawing the triangle to which the hexagons discussed in the first paper degenerate.

I shall take up the different cases in the order in which they are investigated in the first paper and show the way in which the hexagons give rise to inscribed triangles.

Let us take up the following proposition given in my previous paper published in the Bulletin of the Calcutta Mathematical Society.[1]

" P is a point on the side AB of the traingle ABC. The lines PQ, QR, RS, ST, TU and UV are drawn such that PQ is the chord of contact of a circle touching AB, AC; QR the chord of contact of a circle touching CA, CB and so on, the point V must coincide with P.

If $AP=x$,

$AQ=AP=x$,

$\therefore\ CQ=b-x=CR$.

$\therefore\ BR=a-b+x=BS$.

$\therefore\ AS=c-a+b-x=AT$.

If L be the mid-point of PS, we have

$$AL=\tfrac{1}{2}(AP+AS)=\tfrac{1}{2}(c-a+b)=S-a,$$

so that L is the point of contact of the incircle with AB.

[1] Bulletin of the Calcutta Mathematical Society, December, 1928.

Thus if the starting point be taken at the point of contact of the inscribed circle, the starting point will be the middle point of PS and instead of getting the hexagon formed by a set of six points we get the triangle formed by joining the points of contact of the inscribed circle with the sides of the triangle.

If the lines PQ, etc., instead of being cyclic lines be lines parallel to BC, etc.,

by taking $AP=x$, it can be at once shown that

$$AS=c-x.$$

$\therefore$ L being the mid-point of PS as before we have

$$AL=\frac{c}{2}.$$

$\therefore$ L is the mid-point of AB.

Thus the hexagon formed by lines parallel to the sides will degenerate into the triangle formed by the middle points of the sides when the starting point is taken at the middle point of PS.

In the case of antiparallels if $AP=x$

$$AS=2b\cos A-x.$$

$\therefore$ as before $AL=b\cos A$.

$\therefore$ L is the foot of the perpendicular from C on AB.

$\therefore$ The hexagon formed by lines antiparallel to the sides will degenerate into the pedal triangle of the original triangle when the starting point is taken at the middle point of PS.

I shall prove that this will always happen whatever be the nature of the original lines provided the starting point be taken at the mid-point of PS.

For the sake of convenience instead of denoting the second set of points by S, T, and U, I shall denote them by P', Q' and R'.

Let us now take up the problem.

"In a given triangle it is required to inscribe another triangle homological to it of which the sides are to be parallel to three given straight lines."

From the result obtained previously it is seen that in general the problem admits of no solution unless

$$\text{Sin}\,(A+\theta)\ \text{Sin}\,(B+\psi)\ \text{Sin}\,(C+\phi)=\text{Sin}\,\theta\ \text{Sin}\,\phi\ \text{Sin}\,\psi$$

where θ, ϕ and ψ are the angles made by the lines with the sides of the triangle ABC.[1]

Supposing the condition to be satisfied we may find a simple geometrical solution.

The construction in the cases of cyclic lines, parallels and anti-parallels will give us the solution in the perfectly general case as well. If starting from a point P on AB we come back to a point P′ on the same line, then if we start from the middle point of PP′ we shall come back to the starting point by traversing the sides only once.

In other words if the starting point be the mid-point of PP′, the hexagon will reduce to a triangle and we shall get a homological triangle inscribed in ABC of which the sides are parallel to given straight lines.

Let us take a series of points P_1, P_2, etc., on the side AB and starting from these points let us come back to the same line at the points P'_1, P'_2, etc., by traversing the sides of the triangle only once.

Evidently there is one to one correspondence between the two ranges P_1, P_2, etc., and P'_1, P'_2, etc.

Also from our method of traversing the sides as explained in my previous paper, it is easily seen that if we start from P′ we must come back to P.

∴ the two ranges are in involution.

Our problem then will be solved if our starting point be such that P may coincide with P′.

∴ To get the triangle we have to start from one of the double points of this involutionary range.

One double point of this range is at infinity as may be seen in the following way :

Let ξ, η, ζ and ξ', η', ζ' be the points where the line at infinity cuts the sides AB, AC and BC of the triangle ABC and the three lines

$$l_r x + m_r y + n_r z = 0 \quad (r=1,\ 2,\ 3)$$

to which the sides of the homological triangle are to be parallel.

[1] On a class of Transversals cutting the sides of a triangle. Bulletin, Calcutta Mathematical Society, December, 1928, No. 4, Vol. XIX.

Any line parallel to $l_1x+m_1y+n_1z=0$ must pass through ξ'.

If we take ξ as our starting point, the line through ξ parallel to $l_1x+m_1y+n_1z=0$ is the line $\xi\xi'$,
i.e., the line at infinity itself.

$\therefore$ The point Q now takes the position of η.

As before the line through Q parallel to $l_2x+m_2y+n_2z=0$ is the line $\eta\eta'$,

i.e., the line at infinity.

$\therefore$ The point R takes the position of ζ and RP′ becoming $\zeta\zeta'$, *i.e.*, the line at infinity, will cut AB at ξ.

$\therefore$ If we take P at ξ, P′ coincides with P,

$\therefore$ the point at infinity on AB is a double-point of the involution on AB.

Now one focus being at infinity, the other focus will bisect the distance between any pair of conjugate points so that to get the double-point which lies at a finite distance we have only to bisect the line PP′.

Hence we get a simple geometric construction of our problem.

We may use this construction to inscribe a conic in a given triangle of which the chords of contact will be parallel to given straight lines.